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Dijkstra's algorithm for shortest path problem under hesitant fuzzy environment using different operator

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Abstract
Hesitant fuzzy set theory is a branch of fuzzy set theory that uses new measures to address uncertainty in shortest path problems. In this paper, we propose a generalized version of Dijkstra's algorithm from source node to destination node for scenarios where each edge has an associated hesitant fuzzy number as its cost. The Bonferroni mean (BM) is indeed a useful tool in multi-criteria decision-making (MCDM) because it effectively captures the interrelationships among different criteria or arguments. We introduce a modified hesitant fuzzy Dijkstra's algorithm (MHFDA) to address hesitant fuzzy shortest path problems (HFSPP). This algorithm utilizes hesitant fuzzy Bonferroni means (HFBM) and hesitant fuzzy weighted geometric operators (HFWG) to find the solution.

Keywords: Algorithm, operators, solution

1. Introduction
The shortest path problem with fuzzy costs is indeed a well-explored area in fuzzy set theory and fuzzy systems [14]. However, in many real-world scenarios, the cost associated with traveling along a path may not be precisely known and can be represented as fuzzy values. This area of study is crucial for applications where precise data is not available, such as in transportation planning, network design, and decision-making under uncertainty. It sounds like we're referring to a specific approach outlined by researchers [13, 16, 20] to tackle shortest path problems in networks with fuzzy arc lengths using dynamic programming. The dynamic programming method with fuzzy arithmetic to address the uncertainty in arc lengths. By using dynamic programming [5, 14, 16-18, 20, 21, 26, 29-31], the algorithm efficiently computes the shortest path while accommodating the vagueness of fuzzy numbers. This method is particularly useful in real-world applications where precise measurements are difficult to obtain and where costs are inherently uncertain. Variation of Shortest path problem can be found in the paper. The shortest path problem have been solved by many authors by different method with different type fuzzy arc length.

Many researchers [2] have explored aggregation operators for fuzzy numbers, particularly focusing on operations involving Hesitant Fuzzy Numbers (HFNs). The development of aggregation operators for fuzzy numbers and HFNs continues to be an active area of research, with ongoing efforts to refine these operators and explore their applications in different domains. We have a specific paper or study in mind, I can provide more targeted insights based on that work.

In Multi-Criteria Decision-Making (MCDM) problems, aggregation operators are indeed essential tools for combining and synthesizing information from multiple criteria or sources. MCDM problems [15] often involve evaluating and making decisions based on multiple, often conflicting criteria, and aggregation operators help in consolidating these diverse pieces of information into a single, coherent decision. The min and max operators are indeed among the most commonly used operators in fuzzy theory. The min and max operators are advantageous in fuzzy logic for their simplicity and efficiency in calculation. Their extension into a lattice structure provides a robust theoretical framework that supports consistent and well-defined operations in fuzzy systems. This lattice structure not only facilitates the basic operations but also allows for the development of more complex fuzzy logic models and applications. Bonferroni [4] is originally introduced a mean-type aggregation operator which is called the Bonferroni mean.

The Hesitant fuzzy Bonferroni mean (HFBM) was presented by Zhu *et al* [3]. HFBM is a notable aggregation operator used in fuzzy decision-making and multi-criteria decision-making (MCDM) problems. Yager [34] introduced an interpretation of a certain operator and proposed generalizations by substituting it with different types of mean operators. Beliakov [11] conducted a systematic investigation of a family of composed aggregation functions, which generalizes the operator extends the Bonferroni mean to handle hesitant fuzzy information, which is useful when decision-makers express hesitation or uncertainty about their assessments.

Hesitant Fuzzy Sets (HFS) are a generalization of traditional fuzzy sets that are particularly useful for handling situations where there is hesitation or uncertainty in the decision-making process. Their ability to capture the ambiguity and vagueness inherent in human judgments makes them applicable across various fields of science and practical domains. Here's an overview of how hesitant fuzzy sets are being studied and applied in different fields [1]. Using this concept many researchers and practitioners [15, 22-23, 32, 33] have successfully applied hesitant fuzzy sets (HFS) to solve practical problems across a wide range of fields. Tora and Narukawa (2009) and Torra (2010) [36, 37] contributed significantly to the development and formalization of hesitant fuzzy sets and elements. Gau and Buehrer [10] introduced the concepts of vague set. Xia and Xu's [38] work on ranking Hesitant Fuzzy Elements (HFEs) introduces important methodologies for handling and evaluating uncertainty in fuzzy systems.

Edsger W. Dijkstra, a prominent Dutch computer scientist, is well-known for his contributions to algorithm design, particularly for the development of Dijkstra's algorithm [6, 7]. Edsger Dijkstra's algorithm, developed in 1956 and published in 1959, has become a cornerstone of computer science for solving the shortest path problem in weighted graphs. Its efficiency and wide range of applications underscore its importance in both theoretical and practical contexts, making it a fundamental tool in various fields such as networking, robotics, and geographic information systems. Many researchers [8, 12, 16, 19, 24-28, 29, 35] have contributed to the refinement, optimization, and application of this algorithm.

The rest of the paper is organized as follows Section 2 depicts the preliminary concepts of HFSs, HF value, HFNs, score or ranking of HF value and HFBM operator. Section 3 describes the proposed method. Section 4, a numerical example has been solved using the proposed method, Section 5 depicts the results and discussions and Section 6 concludes the paper.

2. Preliminaries Idea

2.1 Definition (Hesitant fuzzy set) (HFS)

Tora and Narukawa [36] introduced the concept of a hesitant fuzzy set as an extension of the traditional fuzzy set to handle cases where there is hesitation or uncertainty in assigning a precise membership value.

Xia and XU, 2011 define HFS as follows-

A hesitant fuzzy set $\hat{A}$ in a universe of discourse $\tilde{X}$ is defined by a collection of fuzzy sets rather than a single fuzzy set. Formally, it can be expressed as:

$$\hat{A} = \left\{\left(\tilde{x}, \{\mu_{\hat{A}_1}(\tilde{x}), \mu_{\hat{A}_2}(\tilde{x}), \ldots, \mu_{\hat{A}_n}(\tilde{x})\}\right) : \tilde{x} \in \tilde{X}\right\} \quad (1)$$

$\{\mu_{\tilde{A}_1}(\tilde{x}), \mu_{\tilde{A}_2}(\tilde{x}), \ldots, \mu_{\tilde{A}_n}(\tilde{x})\}$ Represent a set of possible membership values for an element $\tilde{x}$ in the hesitant fuzzy set $\hat{A}$.

2.2 Definition (Hesitant fuzzy Element) (HFE)

Torra's work extended the concept further by defining the hesitant fuzzy element, which allows for more granular representation of hesitancy in fuzzy logic systems.

A hesitant fuzzy element $\tilde{h}$ in a universe $\tilde{X}$ is characterized by a set of possible values representing the membership degrees of an element $\tilde{x}$ in different fuzzy sets. Formally, a hesitant fuzzy element can be defined as:

$$\tilde{h} = \left\{\left(\tilde{x}, \{\mu_{\hat{A}_1}(\tilde{x}), \mu_{\hat{A}_2}(\tilde{x}), \ldots, \mu_{\hat{A}_n}(\tilde{x})\}\right) : \tilde{x} \in \tilde{X}\right\}$$

Where the set $\{\mu_{\tilde{A}_1}(\tilde{x}), \mu_{\tilde{A}_2}(\tilde{x}), \ldots, \mu_{\tilde{A}_n}(\tilde{x})\}$ contains the possible membership degrees for the element $\tilde{x}$, reflecting the hesitancy in assigning a precise value.

2.3 Score or Ranking of Hesitant Fuzzy Value

A score function [9] of a hesitant fuzzy value (HFV) is introduced by Xia and Xu [38] (2011a), which is represented as follows:

For a HFE $\tilde{h}$, $s(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma}$ (2)

If $\tilde{h}_1$ and $\tilde{h}_2$ are two hesitant fuzzy element then

If $s(\tilde{h}_1) > s(\tilde{h}_2)$ then $\tilde{h}_1$ is superior to $\tilde{h}_2$ denoted by $\tilde{h}_1 > \tilde{h}_2$ (3)

If $s(\tilde{h}_1) < s(\tilde{h}_2)$ then $\tilde{h}_1$ is smaller than $\tilde{h}_2$ denoted by $\tilde{h}_1 < \tilde{h}_2$ (4)

2.4 Some Basic Operaion

Torra and Narukawa (2009) and Torra (2010) is defined some basic operation on HFEs in such way that

Given two HFEs represented by $\tilde{h}_1$ and $\tilde{h}_2$

$$\tilde{h}_1 \bigcup \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\} \quad (5)$$

$$\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \{\tilde{\gamma}_1 + \tilde{\gamma}_2 - \tilde{\gamma}_1\tilde{\gamma}_2\} \quad (6)$$

$$\tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \{\tilde{\gamma}_1\tilde{\gamma}_2\} \quad (7)$$

Where these above notions are useful meaning.

2.5 Bonferroni mean (BM) Operator

The Bonferroni Mean was introduced by the Italian mathematician Carlo Bonferroni in 1950. It is used to aggregate information, particularly in the context of fuzzy sets and multi-criteria decision-making. The Bonferroni Mean is a generalization that can handle various types of aggregation, including those involving fuzzy logic.

Let a_i (i = 1, 2,…, n) be a collection of nonnegative numbers, and p,q≥0 if

$$B^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \quad (8)$$

Then $B^{p,q}$ is called a Bonferroni means.
Where these above notion is useful meaning.

2.6 Hesitant fuzzy Bonferroni mean (HFBM)
The Hesitant Fuzzy Bonferroni Mean (HFBM) introduced by Zhu *et al*. in 2013 is an extension of the Bonferroni Mean to handle hesitant fuzzy information. Hesitant fuzzy sets are used when there is hesitation or uncertainty about the membership degree of elements in a fuzzy set. This type of fuzzy *set al*lows for multiple possible membership values for each element, reflecting more complex decision-making scenarios.

Given that p and q are positive parameters and $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$ where each $\tilde{h}_i$ represents a hesitant fuzzy set, the Hesitant Fuzzy Bonferroni Mean $HFBM^{p,q}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)$ and it is defined by

$$HFBM^{p,q}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(\tilde{h}_i^p \otimes \tilde{h}_j^q)\right)\right)^{\frac{1}{p+q}} \quad (9)$$

Theorem 1: (Zhu *et al*. 2013a). Let p, q>0, and $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$ be a collection of HFEs then aggregated value by using the HFBM is a HFE, and $HFBM^{p,q}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)$ =

$$\bigcup_{\substack{\tilde{\gamma}_{i,j}\in\tilde{\sigma}_{i,j}\\ i\neq j}}\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\tilde{\gamma}_{i,j})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\} \quad (10)$$

Where $\tilde{\sigma}_{i,j} = \tilde{h}_i^p \times \tilde{h}_j^q$ reflects the interrelationship between $\tilde{h}_1$ and $\tilde{h}_2$, i,j =1,2,…, n, i≠j.
Where these above notions are useful meaning.

2.7 Definition 3 (Hesitant fuzzy weighted geometric operator) (HFWG) or (hfwg)
Xia and Xu defined HFWG in 2011 in such way

Let $\tilde{h}_i$ (i= 1,2,…,n) be a collection of HFEs and let HFWG: $\Phi^n \to \Phi$, HFWG is defined by

$$HFWG(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigotimes_{i=1}^{n}\tilde{h}_i^{w_i} =$$

$$\bigcup_{\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2, \ldots, \tilde{\gamma}_n\in\tilde{h}_n}\left\{\prod_{i=1}^{n}\tilde{\gamma}_i^{w_i}\right\} \quad (11)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is called weight vector of $\tilde{h}_i$ (i=1,2,…,n) with $w_i \in [0,1]$, i=1,2,…,n and $\sum_{i=1}^{n} w_i = 1$, where these above notations have usual meaning.

Let $\tilde{h}_1$ ={0.2,0.3} and $\tilde{h}_2$ ={0.4,0.6} be two HFEs, and $w_i = (w_1, w_2)^T = (0.7,0.3)^T$ their weight vector, then we have

$HFWG(\tilde{h}_1, \tilde{h}_2) = \bigcup_{\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2}\{\prod_{i=1}^{2}\tilde{\gamma}_i^{w_i}\}$ = $\{0.2^{0.7}\times0.4^{0.3}, 0.2^{0.7}\times0.6^{0.3}, 0.3^{0.7}\times0.4^{0.3}, 0.3^{0.7}\times 0.6^{0.3}\}$ = {0.2462, 0.2781, 0.3270, 0.3693}

3. The Proposed Method
A connected network (V,E) where V is the set of vertices and E is the set of arcs (or edges) in which 's' is source node and 'e' is the sink node. In the context of finding the shortest path in a network where the cost (time or distance) associated with each arc (i,j)∈E has an associated cost $\tilde{C}_{ij}$, which is represented as Hesitant Fuzzy Elements (HFEs), the problem becomes more complex due to the presence of uncertainty or hesitation in the cost values.

3.1 The Modified Hesitant Fuzzy Dijkstra's Algorithm (MHFDA)
3.2 Input Data
Obtain the network graph G= (V, E).
Gather the hesitant fuzzy weights for each arc.

Compute Aggregated Costs
For each arc (i,j)∈E:

$$\tilde{C}_{ij} = HFBM^{p,q}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)$$

3.3 Graph Transformation
Construct a new graph G′=(V,E′) where each (i,j)∈E has weight $\tilde{C}_{ij}$

3.4 Find Shortest Path
Apply Dijkstra's algorithm on G′ to find the shortest path from's' to e.

3.5 Output
To find the shortest path and the corresponding aggregated cost in a graph
1. To address the problem of calculating the aggregated. Hesitant Fuzzy Value (HFV) for the maximum-cost path or the shortest path in a graph with hesitant fuzzy.
2. The shortest path.

To address the problem where L(x) denotes the label of node x, representing the aggregated hesitant fuzzy value (HFV) for the path from node 's' to node 'x', follow these steps to calculate

3.6 Step 1: Let P =ϕ, where *P* is the set of those nodes which have permanent labels and *T* = {all nodes of the network G}. At first, the permanent label to'*s*' has been assigned as *L(s)* = 1, (initially), '*s*' is the starting node, so definitely it will be present in the shortest path. This is represented by the HFN {1} of the fact that node's' is in the shortest path. Also L(x) = 0 ∀x∈T and x≠'s'.

3.7 Step 2

That node '*v*' in *T* is selected which has the highest score value of its label, called the permanent label of '*v*' (i.e., *L(v))*. Then $P = P \cup \{v\}$ and $T = T - (v)$. Again the node in *T* with highest score value of its label is selected. The new label of a node '*x*' in *T* is given by

$$L(x) = \max \{\text{old } L(x), HFBM (L (v), C_{vx}\} \quad (12)$$

$$L(x) = \max \{\text{old } L(x), HFWG (L (v), C_{vx}\} \quad (13)$$

Where C_{vx} is the HF cost for travelling along the arc $v - x$. Then using the HFBM and HFWG operator of the Eq. (6), (7), (10) and (11) of Section 2.4, 2.6 and 2.7 described in this paper, the aggregated value of *L(v)*and C_{vx} are derived which are also in terms of hesitant fuzzy value. The max function is used for evaluating the maximum of the two HFEs using the Eqs. (2), (3), (4).

It has been assumed that C_{vx}= 0, if there is no edge joining the node 'v' directly to the node 'x'.

3.8 Step 3

STOP. The process of finding the nodes with permanent label is repeated until '*e*' gets a permanent label. The above steps do not actually list the shortest path from the starting node to the terminal node; it only gives the aggregated HFV of the HF cost of travelling the shortest path.

3.9 Step 4

To reconstruct the shortest path, we can indeed work backwards from the terminal node by tracing back through the predecessors. This method is based on the fact that the predecessors dictionary stores the previous node on the shortest path to each node.

3.10 Step 5: End.

In Step 2 for calculating the HFBM (L(v), C_{vx}) or HFWG (L(v), C_{vx}) one has to proceed as follows :

*HFBM (*L(v), C_{vx}) or HFWG (L(v), C_{vx}) are calculated whenever required.

4. Numerical Illustration

Consider a network model G= (V, E) with V = {1, 2, 3, 4, 5, 6} has 6 nodes and E has 10 arcs as shown in the Fig. 1. The Hesitant Fuzzy Cost for travelling along the respective arcs are given in Table 1. The objective is to find the shortest path from node 1 to node 6 such that the total Hesitant Fuzzy (HF) cost of traveling is maximized.

Solution The proposed algorithm MHFDA described in Section 3.1 of this paper has been applied for solving this example.

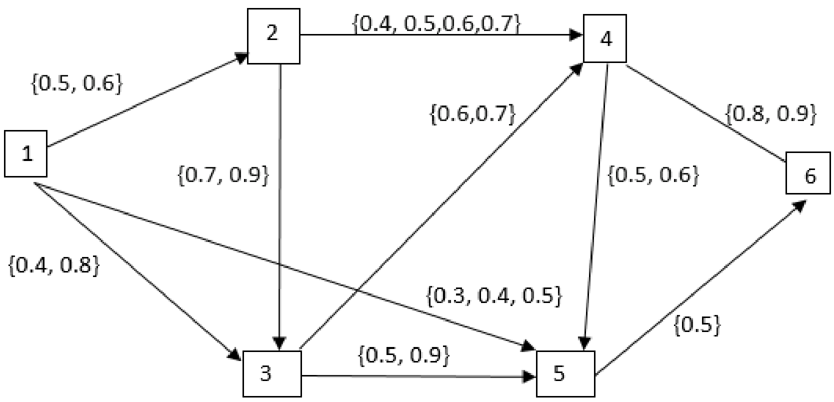

Fig 1: Network Model G=(V,E)G = (V, E)G=(V,E) with Nodes and Hesitant Fuzzy Costs

Table 1: Data for HF Costs for travelling along the respective arcs of the given network

Arcs (i-j)	Hesitant Fuzzy Costs along these arcs c (i, j)
1→2	{0.5,0.6}
1→3	{0.4,0.8}
1→5	{0.3,0.4,0.5}
2→3	{0.7,0.9}
2→4	{0.4,0.5,0.6,0.7}
3→5	{0.5,0.9}
3→4	{0.6,0.7}
4→5	{0.5,0.6}
4→6	{0.8,0.9}
5→6	{0.5}

5. Problem solve by HFWG

5.1 Iteration 1

S= h_1= {1}, C_{12} = {0.5, 0.6}, C_{13} ={0.4, 0.8}

Using the equation (11), we calculate of three hesitant fuzzy numbers h_1 = {1}, C_{12}= {0.5, 0.6} and C_{13} = {0.4, 0.8} we get HFWG (h_1, C_{12}) = {0.5, 0.6}, HFWG (h_1, C_{13})={0.4,0.8}.

Using the equation (2), we calculate the score of two

hesitant fuzzy numbers HFWG (h_1, C_{12}) and HFWG (h_1, C_{13}).
Score of HFWG (h_1, C_{12}) = S(L(x))= 0.55, Score of HFWG (h_1, C_{13})= S(L(x))=0.6.

Table 2: Results of Iteration

Node (x)	L(x)	S(L(x))
2	{0.5,0.6}	0.55
3	**{0.4,0.8}**	**0.6**
4	0	0
5	{0.3,0.4,0.5}	0.4
6	0	0

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration.

5.2 Step 2: S= {1,3}, T= {2,4,,5,6}

5.3 Iteration 2
5.4 Step 1

Table 3: Results of Iteration 2

Node (x)	L(x)	S(L(x))
2	0	0
4	{0.24,0.48,0.28,0.56}	0.39
5	{0.2,0.4,0.36,0.72}	0.42
6	0	0

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration.

5.5 Step 2: S={1,3,5}, T={2,4,6}

5.6 Iteration 3
5.7 Step 1

Table 4: Results of Iteration 3

Node (x)	L(x)	S(L(x))
2	0	0
3	0	0
4	0	0
6	{0.1,0.2,0.18,0.36}	0.21

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration.

5.8 Step 2: S= {1,3,5,6}, T= {2,4}

5.9 Step 3: Since the sink node is 6 so no further processing is needed for this node. *L(6)* = {0.1,0.2,0.18,0.36}, S={1,3,5,6} and s(L(6)) = 0.21 represents the aggregated HFV of the optimum-cost path or the shortest path with respect to the total hesitant fuzzy cost for travelling through the shortest path.

5.10 Step4: The shortest path can be easily constructed as follows:

Pred{6}=5, Pred{5}= 3 Pred{3}=1
So the HF shortest path comes to be *1 → 3* →5→6

5.11 Problem solve by HFBM
5.12 Step 1

Table 5: Results of Iteration 1

Node (x)	L(x)	S(L(x))
2	{0.707, 0.743, 0.775}	0.742
3	**{0.632, 0.808, 0.894}**	**0.778**
4	0	0
5	{0.55,0.55,0.64,0.63,0.67,0.71}	0.63
6	0	0

5.13 Step 2: S= {1, 3}, T= {2, 4, 5, 6}

5.14 Iteration 2
5.15 Step 1

Table 6: Results of Iteration 2

Node (x)	L(x)	S(L(x))
2	0	0
4	{0.616, 0.659, 0.681, 0.642, 0693, 0.72, 0.696, 0.715, 0.726, 0.749,0.732,0.701,0.742, 0.764, 0.665, 0.713, 0.737, 0.752, 0.773, 0.791}	0.681
5	{0.562, 0.601, 0.620, 0.676, 0.754, 0.797, 0.636, 0.653, 0.702, 0.773, 0.812, 0.669, 0.715, 0.782, 0.819, 0.754, 0.811, 0.842, 0.853, 0.877, 0.897}.	0.743
6	0	0

5.16 Step 2: S={1,3,5}, T={2,4,6}

5.17 Iteration 3
5.18 Step 1

Table 7: Results of Iteration 3

Node (x)	L(x)	S(L(x))
2	0	0
4	0	0
6	{0.53, 0.54, 0.544, 0.549, 0.522, 0.554, 0.567, 0.563, 0.566, 0.575, 0.58, 0.582, 0.585, 0.588, 0.589, 0.59, 0.596, 0.598, 0.604, 0.609, 0.553, 0.558, 0.56, 0.564, 0.565, 0.571, 0.575, 0.583, 0.588, 0.593, 0.596, 0.598, 0.603, 0.606, 0.611, 0.616, 0.557, 0.562, 0.564, 0.568, 0.569, 0.575, 0.578, 0.587, 0.591, 0.593, 0.597, 0.599, 0.6, 0.602, 0.607, 0.609, 0.615, 0.619, 0.623, 0.572, 0.575, 0.577, 0.582, 0.585, 0.594, 0.598, 0.606, 0.608, 0.613, 0.616, 0.621, 0.625, 0.5 0.58, 0.586, 0.589, 0.597}	0.52

5.19 Step 2: S= {1,3,5,6}, T={2,4}

5.20 Step 3: Since the sink node is 6 so no further processing is needed for this node. . *L(6)* ={0.53, 0.54, 0.544, 0.549, 0.522, 0.554, 0.567, 0.563, 0.566, 0.575, 0.58, 0.582, 0.585, 0.588, 0.589, 0.59, 0.596, 0.598, 0.604, 0.609, 0.553, 0.558, 0.56, 0.564, 0.565, 0.571, 0.575, 0.583, 0.588, 0.593, 0.596, 0.598, 0.603, 0.606, 0.611, 0.616, 0.557, 0.562, 0.564, 0.568, 0.569, 0.575, 0.578, 0.587, 0.591, 0.593, 0.597, 0.599, 0.6, 0.602, 0.607, 0.609, 0.615, 0.619, 0.623, 0.572, 0.575, 0.577, 0.582, 0.585, 0.594, 0.598, 0.606, 0.608, 0.613, 0.616, 0.621, 0.625, 0.5 0.58, 0.586, 0.589, 0.597}, S={1,3,5,6} and s(L(6)) = 0.52 represents the aggregated HFV of the optimum-cost path or the shortest path with respect to the total hesitant fuzzy cost for travelling through the shortest path.

5.21 Step4: The shortest path can be easily constructed as follows:
Pred{6}=5, Pred{5}= 3 Pred{3}=1
So the HF shortest path comes to be *1* → *3* →5→6.

6. Results and Discussion

The final results can be seen in the solutions of the numerical examples provided above, using the given operator. The results obtained are 0.52 and 0.21 when using the HFBM and HFWG operators, respectively. These results are aggregated as HFN for the path *1* → *3* →5→6 in iteration 3. This implies that the path *1* → *3* →5→6 is the most preferable one for the aggregated HF cost associated with it. Among the two operators, HFWG is the most preferable for finding the shortest distance, compared to the HFBM operator. The main advantage of the proposed method is that, for a given network with fuzzy parameters, it can identify the unique shortest path with the shortest fuzzy distance. Thus, the discrete uncertain knowledge about the cost of traveling along the arcs, represented as HFV, has been mathematically accumulated by the proposed MHFDA method, resulting in a definite solution expressed in terms of both HFV and HFEs. For larger problems, computer programs can be developed based on the proposed methodology. By modifying the well-known Dijkstra's Algorithm to incorporate hesitant fuzzy arc parameters and successfully applying the HFBM operator, this paper introduces a new and efficient heuristic algorithm that addresses of the decision maker. A numerical example demonstrates the effectiveness of the proposed method. In the future, this approach could be extended to address multi-criteria shortest path problems with data represented as HFEs.

7. Conclusion

The Shortest Path Problem (SPP) is a highly significant area of study with applications in various real-life scenarios. This paper introduces a novel and innovative methodology for solving the Shortest Path Problem (SPP) in environments characterized by uncertainty. In practice, the precise values of parameters such as time or cost associated with the arcs of a network may not always be available. To account for uncertainty, fuzzy numbers can be used to represent imprecise or ambiguous values. This paper considers the most general type of fuzzy numbers, namely Hesitant Fuzzy Numbers (HFNs), to represent the uncertain costs associated with traveling through each arc. The HFBM and HFWG operators, a crucial component of HFSs, is utilized to develop the proposed methodology, MHFDA. This type of real-life problem has been efficiently solved using the proposed MHFDA methodology, which successfully applies various existing theories of HFSs. This represents a significant contribution of the paper. In the future, alternative methods could be proposed to address similar problems, and their results compared. Additionally, computer programs could be developed to implement the proposed methodology for large-scale networks.

8. References

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