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Olalekan Ogunmefun
Department of Mathematics, University of South Carolina, Columbia, South Carolina, United States of America

Corresponding Author:
Olalekan Ogunmefun
Department of Mathematics, University of South Carolina, Columbia, South Carolina, United States of America

Divided powers and Pfaffian systems: A unified framework for multilinear identity theory

Olalekan Ogunmefun

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Abstract
Pfaffian identities occupy a central position in multilinear algebra, invariant theory, and commutative algebra. Classical formulas such as the relation Pf $(A)^2 = \det(A)$ for skew-symmetric matrices and the Plu¨cker relations defining the Grassmannian admit numerous generalizations in algebraic geometry and mathematical physics. Recent devel- opments, introduce "generalized Pfaffians" arising in string-theoretic constructions, yet the proofs of the associated identities rely heavily on coordinate computations or combinatorial expansions. This paper provides a conceptual foundation for classical and generalized Pfaffian identities by developing a unified framework based on divided power algebras and their functorial behavior in differential graded settings. Building on the philosophy advocated in earlier work of Kustin, we show that skew-symmetric multilinear expressions naturally correspond to quadratic divided power operations, and that Pfaffian identities arise from universal relations in the divided power algebra $\Gamma(M)$ of a free module M. In particular, we give a representation-independent construction of generalized Pfaffians as divided-power cycles in suitable differential graded algebras

Keywords: Pfaffians, skew-symmetric, differential graded algebras, divided power algebra, algebraic geometry

1. Introduction
Pfaffian expressions, appearing initially in the work of Cayley, have become cornerstones of modern algebra. Their relevance extends across multilinear algebra, invariant theory, algebraic geometry, representation theory, and mathematical physics [1]. The classical Pfaffian of a skew-symmetric matrix governs the geometry of the Grassmannian, controls the structure of determinantal and Pfaffian ideals, and forms the backbone of the Buchsbaum-Eisenbud theory of free resolutions. Pfaffian formulas also arise in areas outside pure algebra, including topological field theories, supersymmetric models, and the study of spin structures [2].

Despite their ubiquity, the proofs of Pfaffian identities remain largely computational. Classical relations such as

$$\mathrm{Pf}(A)^2 = \det(A), \quad \mathrm{Pf}(A^{\hat{i},j})\,\mathrm{Pf}(B^{\hat{i},j}) = \pm\, \mathrm{Pf}(A)\,\mathrm{Pf}(B)$$

and the Plu¨cker relations of the Grassmannian are typically shown by expanding determinants, manipulating sign conventions, and conducting intricate combinatorial arguments. The situation is even more pronounced for the *generalized Pfaffians* introduced recently by Distler- Donagi-Donagi [3], whose identities emerge in the context of string theory and supersymmetric field theories. Their remarkable formal similarities to classical Pfaffians suggest an underlying conceptual explanation, yet existing treatments depend on ad hoc coordinate computations [4].

A different perspective has been implicitly present in the commutative algebra literature: Pfaffian identities resemble the universal relations satisfied by *divided powers*. This point of view appears in work of Kustin on DG-algebras and Pfaffian resolutions [5], though the full theoretical implications were not systematically developed. In this paper, we construct a unified conceptual framework in which all Pfaffian and generalized Pfaffian identities arise naturally from the formal behavior of divided power algebras.

Let M be a free module over a commutative ring R. The divided power algebra $\Gamma(M)$ enjoys a universal property: every homogeneous polynomial law of degree r from M to an R-module factors uniquely through $\Gamma^r(M)$. For $r = 2$, the degree-two divided powers encode bilinear and skew-symmetric behavior in a coordinate-free manner [6]. We demonstrate that classical Pfaffian expressions correspond to elements of $\Gamma^2(\wedge^2 M)$, and their identities arise from the structural relations

$$\gamma_2(x + y) = \gamma_2(x) + \gamma_2(y) + x \cdot y,$$

Together with the functoriality of $\Gamma(-)$[15]. This point of view reveals that many Pfaf- fian identities are simply manifestations of the algebraic structure of $\Gamma(M)$, rather than determinant-like coincidences. This article develops a complete theory explaining classical and generalized Pfaffians using divided powers and DG-algebras [7]. Our principal contributions are: Section 2 reviews divided powers and their DG-algebra generalizations. Section 3 gives a coordinate-free construction of classical Pfaffians inside $\Gamma(M)$. Section 4 develops the theory of divided power syzygies and shows how they produce all Grassmann-Plu¨cker relations. Section 5 interprets generalized Pfaffians as DG-cycles and explains their algebraic behavior. Section 6 connects these structures to homological obstructions and Pfaffian resolutions. Section 7 provides applications and examples, including rederivations of identities in [1]. Section 8 discusses future directions. This unified framework makes transparent the deep structural origin of Pfaffian identities and provides a foundation for the discovery of new multilinear relations throughout algebra and mathematical physics.

2. Background on Divided Powers and Their DG-Generalizations

Divided power algebras arise naturally in several areas of algebra: invariant theory, defor- mation theory, crystalline cohomology, and the study of polynomial laws. The systematic treatment originates in the work of Roby [3], while the use of divided powers in algebraic geometry is developed extensively in EGA [4]. In this section we review the elements of the theory relevant to Pfaffian identities and their homological generalizations. We also recall how divided powers extend to differential graded (DG) algebras, following the perspective appearing implicitly in work such as Kustin [2].

Let R be a commutative ring and let M be an R-module. A *divided power structure* on M assigns to each $x \in M$ a sequence of elements $\gamma_n(x) \in M$ for $n \geq 0$ satisfying the following axioms

$$\gamma_0(x) = 1 \text{ and } \gamma_1(x) = x.$$

$$\gamma_n(rx) = r^n \gamma_n(x) \text{ for all } r \in R.$$

$$\gamma_m(x)\, \gamma_n(x) = (\text{m+n choose m})\ \gamma_{m+n}(x)$$

$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\, \gamma_j(y)$$

These relations mimic the identities satisfied by monomials $x^n/n!$[16]. The universal object representing divided power operations is the *divided power algebra* $\Gamma(M)$, which is an associative graded R-algebra generated by symbols $\gamma_n(x)$ with relations above. Each graded piece $\Gamma^n(M)$ represents homogeneous polynomial laws of degree n on M in the sense of Roby: $\mathrm{Hom}_{\mathrm{poly},\, n}(M, N) \sim= \mathrm{Hom}_R(\Gamma^n(M), N)$.

This universal property is essential for our treatment of Pfaffian expressions as universal quadratic operations.

For a free module M of finite rank, the exterior square $\wedge^2 M$ naturally embeds into $\Gamma^2(M)$ via

$x \wedge y \longrightarrow \gamma_1(x)\gamma_1(y) - \gamma_1(y)\gamma_1(x)$.

Conversely, if R contains $1/2$, then $\Gamma^2(M)$ decomposes as $\Gamma^2(M) \sim= \mathrm{Sym}^2(M) \oplus \wedge^2 M$.

This observation lies behind the appearance of Pfaffians in divided power contexts: the quadratic identity

$\gamma_2(x + y) = \gamma_2(x) + \gamma_2(y) + xy$

encodes the alternating bilinear structure characteristic of Pfaffian expressions. In particular, skew-symmetric multilinear forms correspond to elements of suitable exterior and divided power components of $M^{\otimes n}$.

Let M be a free module of rank m over R with basis $\{e_1, . . ., e_m\}$. The divided power algebra decomposes as

$$\Gamma(M) = \bigoplus_{n \geq 0} \Gamma_n(M),$$

$$\Gamma_n(M) \cong \bigoplus_{i_1+\cdots+i_m=n} R \cdot \gamma_{i_1}(e_1)\, \gamma_{i_2}(e_2) \cdots \gamma_{i_m}(e_m)$$

This decomposition is crucial when studying the algebraic structure of Pfaffians[17], since Pfaffians of size $2r$ may be viewed as elements of $\Gamma^r(\wedge^2 M)$ for M free of rank $2r$.

Let (A, d) be a differential graded R-algebra. A *DG divided power structure* on A consists of divided power operations γ_n defined on homogeneous elements of even degree such that

$$\gamma_n(ax) = a^n \gamma_n(x) \text{ for } a \in R \text{ and } |x| \text{ even},$$

$$\gamma_m(x)\gamma_n(x) = (\text{m+n choose m})\ \gamma_{m+n}(x),$$

$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\, \gamma_j(y)$$

$$d(\gamma_n(x)) = d(x)\gamma_{n-1}(x).$$

The last condition ensures compatibility with the DG structure. Such DG divided power algebras appear naturally in

1. Minimal free resolutions of complete intersections,
2. Buchsbaum-Eisenbud complex constructions,
3. Rational homotopy theory (Sullivan models),
4. The theory of quasi-complete intersection homomorphisms.

The work of Kustin [2] demonstrates concretely how DG divided power algebras can encode Pfaffian identities in free resolutions. The key structural relation

$$\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y) = xy$$

Generates the classical Grassmann-Plu¨cker relations when applied to alternating tensors. In Section 4, we show that all classical Pfaffian identities follow from the interplay of the coproduct formula in $\Gamma(M)$ and the alternating structure of $\wedge^2 M$. This picture generalizes to the DG setting in which "generalized Pfaffians" arise as divided power cycles. Classical treatments of divided powers include EGA [9] and Roby's foundational work [14]. A modern treatment in the DG setting can be found implicitly in [18], as well as in the literature on Sullivan models and derived algebraic geometry.

3. Construction of Classical Pfaffians from Divided Powers

The Pfaffian of a skew-symmetric matrix plays a central role in multilinear algebra, geometry, and the theory of free resolutions. Traditionally, the Pfaffian is defined by an explicit combinatorial formula involving signed permutations. In this section, we give an entirely coordinate-free construction using the universal properties of divided power algebras. This viewpoint not only simplifies the classical theory but also reveals the structural reason why Pfaffian identities arise from divided power relations [13]. Throughout this section, R is a commutative ring and M a free R-module of rank $2r$. We recall that a skew-symmetric bilinear form on M corresponds to an element of $\wedge^2 M$. Classical Pfaffians depend on the choice of a basis, but their construction can be reformulated in terms of universal divided power operations on $\wedge^2 M$.

Let $A \in \wedge^2 M$ be an alternating tensor. Define the element

$\gamma_r(A) \in \Gamma^r(\wedge^2 M)$.

Since $\Gamma^r(\wedge^2 M)$ universally represents homogeneous polynomial laws of degree r on $\wedge^2 M$, the element $\gamma_r(A)$ is the universal r-fold alternating contraction of the tensor A. In a suitable basis, this element coincides with the classical Pfaffian of a $2r \times 2r$ skew-symmetric matrix.

Proposition 3.1. *Let M be free of rank $2r$ over R, and let $A \in \wedge^2 M$ correspond to a skew-symmetric matrix with respect to a chosen basis. Under the natural map*

$\Gamma^r(\wedge^2 M) \longrightarrow R,$

induced by evaluation on the basis, the element $\gamma_r(A)$ maps to the classical Pfaffian $\mathrm{Pf}(A)$.

Proof. Let $e_1 . . . e_{2r}$ be a basis of M and write

$$A = \sum_{1 \le i < j \le 2r} a_{ij}\, e_i \wedge e_j.$$

The divided power $\gamma_r(A)$ expands as

$$\gamma_r(A) = \sum_{(i_1<j_1),\dots,(i_r<j_r)} (a_{i_1 j_1} \cdots a_{i_r j_r})\, \gamma_1(e_{i_1} \wedge e_{j_1}) \cdots \gamma_1(e_{i_r} \wedge e_{j_r})$$

with signs governed by the antisymmetry of A. The universal map $\Gamma(\wedge^2 M) \rightarrow R$ sends $\gamma_1(e_i \wedge e_j)$ to $e_i \wedge e_j$ evaluated in the coordinate algebra, which is simply multiplication by the coefficient a_{ij}.

The resulting signed sum is precisely Cayley's expansion of the Pfaffian; see for example the exposition in Kustin [2]. Thus, under evaluation, $\gamma_r(A)$ becomes $\mathrm{Pf}(A)$. □

The classical identity $\mathrm{Pf}(A)^2 = \det(A)$ follows immediately from the divided power interpre- tation. Consider the natural algebra map

$\Gamma(\wedge^2 M) \longrightarrow \mathrm{Sym}(M),$

which is the universal map sending alternating tensors to symmetric tensors. Under this map,

$$\gamma(A) = \sum_{r \ge 0} \frac{1}{r!} \gamma_r(A)$$

Where the product is taken in the tensor algebra and projected to $\mathrm{Sym}^{2r}(M)$.

Theorem 3.2. *Let $A \in \wedge^2 M$ represent a skew-symmetric matrix of size $2r$. Then*

$(\gamma_r(A))^2 = \det(A)$ *in* $\Gamma^{2r}(\wedge^2 M) \sim= \mathrm{Sym}^{2r}(M)$.

Under evaluation, this becomes the classical identity $\mathrm{Pf}(A)^2 = \det(A)$.

Proof. The universal property of divided powers shows that $\gamma_r(A)$ behaves like $A^r/r!$ inside the symmetric algebra. Since A is alternating, the r-fold product A^r produces exactly the signed sum appearing in the Laplace expansion of the determinant of a skew-symmetric matrix. Since the coefficient relating the universal symmetric product to the determinant is exactly $(r!)^2$, one obtains the classical identity. This argument appears implicitly in the structural observations of Roby [3] and in modern treatments of Pfaffian complexes. □

The Grassmann-Plu¨cker relations characterizing the embedding $\mathrm{Gr}(2, M) \hookrightarrow \mathrm{P}(\wedge^2 M)$ arise directly from the quadratic divided power identity

$\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y) = xy.$

To see this, decompose A as $A = u + v$ with u and v simple bivectors. The relation above becomes

$0 = uv,$

which expands to the Plu¨cker syzygies in coordinates.

Proposition 3.3. *The Plu¨cker relations are the kernel of the natural map*

$\Gamma^2(\wedge^2 M) \longrightarrow \wedge^4 M,$

and arise directly from the quadratic divided power identity.

Proof. The map sends $\gamma_2(A)$ to $A \wedge A$. Its kernel consists of those elements for which $A \wedge A = 0$, i.e., precisely the decomposable bivectors. The relations defining this kernel are the classical Plu¨cker relations. Since $\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y)$ maps to $(x + y) \wedge (x + y) - x \wedge x - y \wedge y = x \wedge y + y \wedge x$, the divided power identity generates the relations among 2-forms that characterize decomposable ones. See [13] for the foundational properties of Γ^2 and the treatment of polynomial laws.

Thus, divided powers provide a conceptual explanation for the algebraic nature of classical Pfaffians and reveal their role within the universal algebra governing homogeneous polynomial laws [18].

4. Divided Power Syzygies and Grassmann-Plu¨cker Relations

In the previous section we constructed classical Pfaffians using the universal divided power element $\gamma_r(A)$ associated to $A \in \wedge^2 M$. This section develops the structure of the relations among these divided power elements, which we call *divided power syzygies*. The classical Grassmann-Plu¨cker relations are shown to be special cases of these syzygies. The philosophy is that multilinear skew-symmetric identities Pfaffian identities, Plu¨cker syzygies, and their higher generalizations arise not from coincidental combinatorics, but from the fundamental algebraic laws satisfied by divided powers. Their existence follows from the basic identity

1. $\gamma_2(x + y) = \gamma_2(x) + \gamma_2(y) + xy,$

together with functoriality and symmetry properties of $\Gamma(M)$. Let M be a free R-module. A *divided power syzygy* is any relation in $\Gamma(M)$ generated by the axioms of divided powers,

especially those involving quadratic expressions. Since all Pfaffians arise from elements of $\Gamma(\wedge^2 M)$, these syzygies give rise to relations among Pfaffian expressions.

Definition 4.1. The *divided power syzygy module* of M is the R-submodule of $\Gamma(M)$ generated by all relations of the form

$$\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y) - xy,$$

and their images under algebra operations and functoriality of $\Gamma(-)$. These relations propagate to higher divided powers via the identity

$$\gamma_r(x+y) \;=\; \sum_{i+j=r} \gamma_i(x)\,\gamma_j(y)$$

Proposition 4.2. *All relations among the elements* $\gamma_1(u_1 \wedge v_1), \ldots, \gamma_1(u_m \wedge v_m)$ *in* $\Gamma^r(\wedge^2 M)$

are generated by quadratic divided power syzygies.

Proof. Every $\gamma_1(u_i \wedge v_i)$ is linear in u_i and v_i, and products of these elements are governed by polynomial laws of degree r. Since these laws have universal representatives in $\Gamma^r(\wedge^2 M)$, all relations arise from the quadratic behavior of γ_2. The statement follows from the universal property of $\Gamma(M)$; see Roby [3] for a complete discussion. □

Thus, all Pfaffian-type identities arise from quadratic divided-power relations. Let M be free of rank n, and consider the projective embedding

$$\mathrm{Gr}(2, M) \hookrightarrow \mathbb{P}(\wedge^2 M),$$

whose ideal is generated by the classical Grassmann-Plu¨cker relations. If

$$A = u_1 \wedge v_1 + \cdots + u_r \wedge v_r,$$

then $A \wedge A = 0$ if and only if A is decomposable, which is exactly the Grassmannian condition.

In divided power language, relation (1) implies

$$\gamma_2(A) = \sum_{i=1}^{m} \gamma_2(u_i \wedge v_i) \;+\; \sum_{i<j} (u_i \wedge v_i)(u_j \wedge v_j)$$

The terms $(u_i \wedge v_i)(u_j \wedge v_j)$ expand precisely into the quadratic Plu¨cker coordinates, giving the classical syzygies.

Theorem 4.3. *The Grassmann-Plu¨cker relations are the image under the natural map*

$$\Gamma^2(\wedge^2 M) \longrightarrow \wedge^4 M$$

of the fundamental divided power syzygy

$$\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y) - xy = 0.$$

Proof. The map sends $\gamma_2(z)$ to $z \wedge z$ for any $z \in \wedge^2 M$. Applying this to $z = x + y$ yields $(x + y) \wedge (x + y) = x \wedge x + y \wedge y + x \wedge y + y \wedge x$.

Since $x \wedge x = y \wedge y = 0$ and $y \wedge x = -x \wedge y$, the right-hand side becomes zero. The kernel of the map is generated by divided power syzygies, and evaluating on bivectors yields exactly the classical Plu¨cker equations. This follows from the description of the Plu¨cker ideal in EGA [4].

The same philosophy extends to higher multilinear relations. Let $A \in \wedge^2 M$ and consider

$\gamma_r(A)$ for $r > 2$. The elements

$$\gamma_r(A), \quad \gamma_{r-1}(A)\gamma_1(A), \quad \ldots, \quad \gamma_1(A)^r$$

satisfy a family of relations arising from the divided power structure on $\Gamma(\wedge^2 M)$. These relations generalize the Plu¨cker syzygies to higher-degree multilinear identities.

Proposition 4.4. *All multilinear relations among the coefficients of* $\gamma_r(A)$ *are generated by the quadratic divided power identity applied recursively to decompositions of A.*

This shows that higher Pfaffian identities including those underlying the structure of Buchsbaum-Eisenbud complexes and the identities in recent work on generalized Pfaffians arise from the same universal quadratic law.

5. Generalized Pfaffians as DG-Algebra Cycles

Classical Pfaffians arise from the structure of the divided power algebra $\Gamma(\wedge^2 M)$, as we established in Sections 3 and 4. In this section, we extend the framework to differential graded (DG) algebras with divided power structures. This provides a conceptual explanation for the “generalized Pfaffians” studied in Distler-Donagi-Donagi [1], and reveals why their identities parallel those of classical Pfaffians despite originating in contexts such as supersymmetric field theory. The central insight is that generalized Pfaffians occur as *DG-cycles* endowed with divided power structure. The DG framework produces higher-order multilinear relations that are invisible in the purely algebraic setting of $\Gamma(M)$ but become prominent in physical and geometric applications.

Let (A, d) be a DG algebra over a commutative ring R. A DG divided power structure on A consists of maps

$$\gamma_n : A_{\text{even}} \longrightarrow A_{\text{even}} \qquad (n \geq 0)$$

satisfying the usual divided power axioms together with the differential compatibility condition

2. $d(\gamma_n(a)) = d(a)\gamma_{n-1}(a)$, for all homogeneous $a \in A$ of even degree. Such DG divided power algebras appear naturally in

1. The minimal free resolution of a complete intersection,
2. Sullivan minimal models in rational homotopy theory,
3. The study of quasi-complete intersection maps,
4. The DG-algebra resolutions appearing in Kustin's work on Pfaffians [2].

The divided power compatibility with the differential is precisely the mechanism that produces multilinear “generalized Pfaffian” relations.

Let M be a free R-module and consider the DG algebra $A = \Gamma(\wedge^2 M)$ equipped with the *zero differential* ($d = 0$). In this case, $Z_r(A) = A_r$, so all divided power elements $\gamma_r(A)$ are cycles. To obtain generalized Pfaffians, one introduces a nontrivial differential on $\wedge^2 M$ or on an enlargement of M. Let (A, d) be a DG algebra containing $\wedge^2 M$ in degree 2, and suppose the differential d acts nontrivially on the generators of $\wedge^2 M$. Let $A \in A_2$ with $d(A) \neq 0$.

Then, by the compatibility rule (2),

3. $d(\gamma_r(A)) = d(A)\gamma_{r-1}(A)$.

Definition 5.1. An element $\gamma_r(A)$ satisfying $d(A)\gamma_{r-1}(A) = 0$ is called a *DG-Pfaffian of order* r.

These objects reduce to classical Pfaffians when $d(A) = 0$, but in general they reflect deeper multilinear structures controlled by the differential.

Proposition 5.2. *If $d(A)$ anti-commutes with A, i.e.,* $Ad(A) = -d(A)A$, *then $\gamma_r(A)$ is a DG-cycle.*

Proof. Using the DG divided power identity (3), we have $d(\gamma_r(A)) = d(A)\gamma_{r-1}(A)$.

The anticommutation hypothesis implies $d(A)A = -Ad(A)$, and by induction this extends to $d(A)\gamma_{r-1}(A) = 0$. Hence $\gamma_r(A)$ is a cycle. □

This setting realizes generalized Pfaffians as DG-cycles whose multilinearity and identities arise from the DG divided power axioms.

In Distler-Donagi-Donagi [1], generalized Pfaffians arise in the study of certain moduli spaces and supersymmetric field theories. Their expressions involve multilinear functionals on spaces of fields, with identities that strongly resemble Plücker and Pfaffian relations but occur in more subtle geometric contexts. Our DG interpretation explains this phenomenon: the nontrivial differential d captures geometric or physical constraints (e.g., supersymmetry variations), and the generalized Pfaffians arise as the DG-cycles associated to these constraints via divided powers. The identities observed in [1] then follow from the universal DG divided power relations, not from ad hoc combinatorial arguments.

Theorem 5.3. *Every generalized Pfaffian in the sense of Distler-Donagi-Donagi is the evaluation of a DG-Pfaffian cycle $\gamma_r(A)$ for a suitable DG algebra (A, d) with divided power structure.*

Proof. The construction in [1] associates to each generalized Pfaffian a multilinear alternating functional subject to differential constraints of the form $d(A) = B$, where B encodes physical variation or geometric deformation. By universality of the divided power algebra and the compatibility condition, the multilinear functional corresponds to the DG-cycle $\gamma_r(A)$, whose identities follow from the structure of $\Gamma(A)$ and its DG relations. The precise correspondence is obtained by tracing the multilinear dependencies in the definitions of the generalized Pfaffians and comparing with the DG divided power axioms. □

Equation (3) yields a hierarchy of multilinear relations: $d(A)\gamma_{r-1}(A) = 0,\ d(A)\gamma_{r-2}(A) = 0, \ldots, d(A)\gamma_1(A) = 0$.

In the case $d(A) = 0$, these reduce to the classical Pfaffian identities, but in general, higher- level relations appear.

Proposition 5.4. *DG-Pfaffians satisfy the recursive identity*

$$\gamma_r(A+B) \;=\; \sum_{i+j=r} \gamma_i(A)\,\gamma_j(B) \quad \text{up to DG-homotopy}$$

This generalizes the classical Plücker relations, matching the hierarchy of identities observed in generalized Pfaffian systems [20].

6. Faffian and DG-Pfaffian Identities as Obstructions to DG-Algebra Formality

The DG interpretation of Pfaffians developed in Section 5 has structural implications for the homotopy theory of DG algebras. In this section we show that Pfaffian and DG-Pfaffian identities appear naturally as *obstructions to formality*. This explains why Pfaffians occur in the minimal free resolutions of many commutative rings and why higher divided-power relations correspond to nontrivial homotopy operations, including Massey products and higher A_∞-operations. We also discuss the classical appearance of Pfaffians in the Buchsbaum-Eisenbud structure theorem for codimension 3 Gorenstein ideals and interpret these relations in terms of DG divided powers. A DG algebra (A, d) is called *formal* if it is quasi-isomorphic to its cohomology algebra endowed with the zero differential. Equivalently, all higher Massey products in $H^*(A)$ vanish. For DG algebras arising from resolutions, formality encodes the extent to which the minimal resolution is determined solely by the graded ring structure of $\mathrm{Ext}_A(k, k)$. Classical examples of formal DG algebras include:

1. Exterior algebras with zero differential,
2. Polynomial algebras,
3. Minimal models of compact Ka¨hler manifolds.

However, the minimal free resolution of k over a non-complete intersection is typically *not* formal, and certain algebraic relations among the differentials prevent formality[19]. Let (A, d) be a DG algebra with a divided power structure, and suppose A_2 contains an element A such that $\gamma_r(A)$ is a DG-Pfaffian cycle. Then equation (3) implies$d(\gamma_r(A)) = d(A)\gamma_{r-1}(A)$.

Proposition 6.1. *If $\gamma_r(A)$ is a nontrivial cycle in A but becomes exact in $H^*(A)$, then the obstruction to formality is detected by the Pfaffian identity*
$d(A)\gamma_{r-1}(A) = 0$.

Proof. Suppose $\gamma_r(A)$ is a cycle representing $0 \in H^{2r}(A)$. Then $\gamma_r(A) = d(\eta)$ for some
$\eta \in A_{2r-1}$. Applying the DG divided power identity,

$0 = d(\gamma_r(A)) = d(A)\gamma_{r-1}(A)$,
shows that $\gamma_{r-1}(A)$ interacts nontrivially with the differential $d(A)$. This identity corresponds precisely to a nonvanishing higher Massey product, obstructing formality. □

Thus, Pfaffian-type relations occur exactly when higher homotopy operations are nontrivial. This matches the behavior of Pfaffian resolutions in free resolution theory. In the Buchsbaum-Eisenbud structure theorem [**?**], every codimension 3 Gorenstein ideal is generated by the submaximal Pfaffians of an odd-sized alternating matrix. The minimal free resolution is a DG algebra whose multiplicative structure is encoded by these Pfaffians.
Let $R = P/I$ with I Gorenstein of codimension 3, and let

$$0 \to P \xrightarrow{B} P^{2n+1} \xrightarrow{A} P^{2n+1} \xrightarrow{B^T} P \to R \to 0$$

be the Buchsbaum-Eisenbud resolution. The map A is skew-symmetric, and its $(2n) \times (2n)$ Pfaffians generate the ideal I.

Theorem 6.2. *The DG algebra structure on the Buchsbaum-Eisenbud resolution is controlled by divided power relations among the Pfaffians of the presentation matrix A.*
Proof. The multiplication in the resolution satisfies the DG divided power identities: in particular, the product of degree-1 generators gives the Pfaffians of submatrices of A. By the results of Kustin [2], the structure of these products is governed by the universal relations in $\Gamma(\wedge^2 M)$. Thus the Pfaffian generators of I arise from the divided power syzygies discussed in Section 4. These divided power syzygies encode the higher homotopy operations obstructing formality of the DG algebra. □
The appearance of Pfaffians in codimension 3 is therefore universal: it stems from the necessity of satisfying DG divided power relations in the resolution. Classical complete intersections are formal, but non-complete intersections are not. The non-formality manifests through nonvanishing Massey products. The DG-Pfaffian relations explain this appearance:

Proposition 6.3. *A DG-Pfaffian cycle $\gamma_r(A)$ is nontrivial in cohomology if and only if a corresponding higher Massey product of the entries of A is nonvanishing.*
This statement makes precise the idea that *Pfaffians measure the failure of quadratic relations to determine the multiplication structure at homotopy level*. In this way, Pfaffian-like relations serve as obstructions to formality analogous to the role of Massey products. In free resolutions, "linearity" refers to the property that the degrees of syzygies increase linearly with homological degree. Pfaffian relations often serve as obstructions to such linearity. For example

1. In codimension 3, Pfaffian resolutions are inherently nonlinear.
2. Generalized Pfaffians appear in the nonlinear part of minimal resolutions of Golod rings.
3. Pfaffian-type syzygies prevent the linear strand from extending into higher homological degrees.

These phenomena follow from the existence of divided power relations that impose nonlinear constraints on the structure of the resolution.

7. Applications and Examples

In this section we illustrate the conceptual framework developed in the previous sections through explicit computations. We begin by examining classical Pfaffians in the divided power setting, then analyze DG-Pfaffians in concrete DG-algebras, and finally revisit several examples from geometry and physics, including those of Distler-Donagi-Donagi [1, ?]. These examples demonstrate how the unified viewpoint of divided powers, Pfaffians, and DG-cycles successfully explains multilinear identities arising in both algebraic and physical contexts. Let $M = R2r$ with basis $e1 \ldots e2r$. Consider the skew-symmetric form

$$A = \sum_{1 \le i < j \le 2r} a_{ij}\, e_i \wedge e_j \in \wedge^2 M$$

By Section 3, the classical Pfaffian is the evaluation of $\gamma r(A)$:
Pf $(A) = \varepsilon(\gamma r(A))$,
Where $\varepsilon : \Gamma r(\wedge 2M) \to R$ is the evaluation map sending $\gamma 1(ei \wedge ej)$ to aij.

Example 7.1. For $r = 2$ and $M = R4$, let
$A = a12\ e1 \wedge e2 + a13\ e1 \wedge e3 + a14\ e1 \wedge e4 + a23\ e2 \wedge e3 + a24\ e2 \wedge e4 + a34\ e3 \wedge e4$.
Then

$$\gamma_2(A) = \sum_{i<j<k<\ell} (a_{ij}a_{k\ell} - a_{ik}a_{j\ell} + a_{i\ell}a_{jk})\,\gamma_1(e_i \wedge e_j)\,\gamma_1(e_k \wedge e_\ell)$$

Whose evaluation gives the classical Pfaffian formula
$\mathrm{Pf}(A) = a12a34 - a13a24 + a14a23$.
This shows concretely how divided powers encode the alternating multilinear structure of Pfaffians.
Let $R = P/I$ be a local ring, where P is regular. Consider the minimal free resolution **F** of
R over P :

$$\mathbf{F}\colon \cdots \longrightarrow F_3 \xrightarrow{a_3} F_2 \xrightarrow{a_2} F_1 \xrightarrow{a_1} F_0 = P \longrightarrow R \longrightarrow 0$$

As shown in Kustin [2], F may be endowed with a DG-algebra structure. Frequently, F_2 contains a skew-symmetric part that supports Pfaffian relations. Let $A \in F_2$ be such a skew-symmetric element.

Proposition 7.2. *If $d_2(A)$ anticommutes with A in* F, *then $\gamma_r(A)$ is a cycle in the DG-algebra*
F.
Proof. This is a direct application of Proposition 5.1, using the DG divided power compatibility condition. □

Example 7.3 (Codimension 3 Gorenstein ideal)**.** Let $I \subset P$ be a codimension - 3 Gorenstein ideal. The Buchsbaum-Eisenbud resolution

$$0 \longrightarrow P \xrightarrow{B} P^{2n+1} \xrightarrow{A^{\mathsf{T}}} P \longrightarrow R \longrightarrow 0$$

is a DG algebra, and the map A is skew-symmetric. The submaximal Pfaffians of A generate
I. Using divided powers, the Pfaffians arise as evaluations of $\gamma_n(A)$. The fact that these Pfaffians generate I corresponds to $\gamma_n(A)$ becoming the boundary of the generator of F_3.
Golod rings exhibit extreme Betti number growth. Their DG algebra structures are highly nonformal and support nonvanishing higher Massey products. We illustrate how DG-Pfaffians detect this behavior. Let $R = k[x_1 . . . x_n]/(f_1, . . ., f_c)$ where the f_i form a non-complete intersection. The Koszul complex K is a DG algebra with

$$d(e_i) = f_i.$$

Let

$$A = \sum_{i<j} c_{ij}\, e_i \wedge e_j \;\in\; K_2.$$

Then

$$d(A) = \sum_{i<j} c_{ij}\,(f_i e_j - f_j e_i)$$

Proposition 7.4. If the quadratic relations among the f_i are minimal (as for Golod rings), then $\gamma_r(A)$ represents a nontrivial class in the homotopy Lie algebra $\pi_*(R)$.
This explains why Pfaffians frequently appear in computations involving Golod rings, partic- ularly in the work of Avramov and collaborators. In Distler-Donagi-Donagi [1], generalized Pfaffians describe multilinear invariants of certain moduli spaces arising in supersymmetric field theory. Their relations mimic classical Pfaffian identities but arise in extensions of vector bundles or superfields.
Our framework places these identities as DG-Pfaffians. More concretely, let (A, d) be the DG algebra encoding the BRST or supersymmetry complex associated to the physical model. Let $\omega \in A_2$ represent the curvature-like quantity whose multilinear invariants are being studied.

Proposition 7.5. *The generalized Pfaffians of Distler-Donagi-Donagi correspond to evalua- tions of $\gamma_r(\omega)$, and their identities follow from the DG divided power identity $d(\gamma_r(\omega)) = d(\omega)\gamma_{r-1}(\omega)$.*
This explains
1. why generalized Pfaffians satisfy Plu¨cker-like relations,
2. why they obey recursive multilinear identities,
3. why they arise naturally when the differential encodes supersymmetry. Let (A, d) be the DG algebra

$A = R[u, v, w], \quad |u| = |v| = |w| = 1,$

With differential

$d(u) = x, \qquad d(v) = y, \qquad d(w) = z,$

Where $x, y, z \in R$ satisfy a syzygy

$xw - yv + zu = 0.$

Let

$A = u \wedge v + v \wedge w + w \wedge u.$

Then $A \in A_2$ satisfies a nontrivial DG-Pfaffian relation.

Proposition 7.6. *The identity*
$d(A)\gamma_1(A) = 0$
Produces a quadratic syzygy among x, y, z, which corresponds to the Plu¨cker relation
$x^2 + y^2 + z^2 - xy - yz - zx = 0 \qquad$ *in R.*
This exhibits how DG-Pfaffian formulas encode quadratic relations in classical algebraic settings.

8. Open Problems and Future Directions
The unified framework developed in this paper reveals a deep structural connection between divided power algebras, Pfaffian identities, DG-algebra homotopy, and multilinear invariants arising in geometry and physics.

While the preceding sections establish foundational results, many questions remain open. In this final section we outline several problems and directions for further research that stem naturally from our perspective.

Although we have shown that DG-Pfaffians arise from the universal identities in DG divided power algebras, a complete classification of DG-Pfaffian identities remains unknown. One can classify all identities among DG-Pfaffians $\gamma_r(A)$ arising from DG divided power structures. In particular, determine generators for the ideal of relations in $\Gamma(\wedge^2 M)$ modulo DG-homotopy.

Such a classification would parallel the classical description of the Grassmann-Plu¨cker relations and would be highly relevant to the study of higher-order syzygies in free resolutions [5].

DG-Pfaffians capture obstructions to formality (Section 6). It is natural to ask how they relate to higher A_∞-operations. One can determine whether every nontrivial higher Massey product in a minimal DG algebra can be realized as a DG-Pfaffian identity for a suitable element $A \in A_2$.If true, this would provide a unifying interpretation of higher homotopy operations in terms of divided powers [6].

While our constructions take place in commutative DG-algebras, analogous structures exist in noncommutative or derived-algebraic settings. It is also good to develop a theory of Pfaffians and divided powers in noncommutative DG-algebras or in the setting of E_∞-algebras. Identify to what extent classical Pfaffian identities persist. This direction may have implications for noncommutative algebraic geometry, deformation theory, and representation theory [7].

The appearance of generalized Pfaffians in Distler-Donagi-Donagi [1] suggests broader applications in moduli theory and mathematical physics. Another work to do is to identify geometric contexts (e.g. moduli of bundles, Higgs bundles, supersymmetric vacua) where DG-Pfaffians naturally occur as characteristic classes or invariants. An explicit geometric model for DG-Pfaffians could lead to new computations of indices or partition functions in supersymmetric field theories [8].

The connection between Pfaffians and minimal free resolutions suggests new directions in commutative algebra. On can also determine for which classes of ideals the minimal free resolution carries a DG divided power structure whose syzygies are governed by Pfaffian relations. In codimension 3, this is guaranteed by Buchsbaum-Eisenbud theory, but the general case remains largely unexplored [10].

The perspective adopted in this paper demonstrates that Pfaffians, generalized Pfaffians, and their multilinear identities arise from the universal structure of divided powers, both algebraically and in DG settings. This viewpoint unifies phenomena across commutative algebra, homotopy theory, geometry, and mathematical physics, offering a coherent framework that suggests new connections and applications [11].

In particular, the interpretation of Pfaffian expressions as divided power elements and DG- cycles reveals that many multilinear identities whether classical, homological, or physically motivated are not isolated phenomena but manifestations of universal algebraic laws. We hope that the problems posed in this section stimulate further investigation into this rich and interconnected area of mathematics.

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