International Journal of Circuit, Computing and Networking

E-ISSN: 2707-5931
P-ISSN: 2707-5923
IJCCN 2023; 4(2): 01-12
Received: 03-05-2023
Accepted: 05-06-2023

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Mathematical modelling of a nonlinear heat conduction of the human head with heat loss and Newtonian heating

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Abstract
Temperature profiles within the human head are highly influenced by the geometry and the rate of metabolic heat production. In this research work, we provide an approximate solution to a nonlinear singular boundary value problem modelling the distribution of heat sources in the human head and their dependence on environmental temperature. Our analysis is based on Runge-Kutta method of order four alongside with shooting technique. Graphical results are presented and discussed quantitatively. The model serves as a tool for extrapolating human head temperature profiles. This research work considers graphically the effects of thermogenesis parameter and Biot number on a nonlinear heat condition model of the human head. The results show that the temperature of the human head is influenced by the thermogenesis slope parameters, thermogenesis heat production parameter and the Biot numbers. The results obtained follow previous research works in the literature. The entire work is based on the theorem for the existence and uniqueness of solution with imposed conditions.

Keywords: Mathematical modelling, nonlinear heat conduction, human head and Newtonian heating

Introduction

The heat transfer in human body is an important and vibrant field, which helps in analyzing the human heat stress in various atmospheric conditions. The two systems namely the active and passive systems constitute the interactive thermoregulatory set-up of human body (Kamangar, Khan, and Badruddin, 2019) [2]. The active system controls the human body temperature at certain levels and also predicts regulatory responses such as shivering, vasomotion and sweating. The passive system simulates heat transfer with the surroundings. According to Kamangar, *et al.* (2019) [2], the human body behaves as heat engine and acts as open system thermodynamically. The bio heat is produced by the various chemical reactions taking place in the human body in turn provides energy for the systems of the body. In order to maintain healthy bodily functions, the human body must maintain a constant internal temperature. Hence the regulation of the internal heat with the surroundings would assist in this process. However, the various metabolic activities (oxidation of food elements) in the human body also assist to maintain the required body temperature. Heat generated in the human body by metabolism is dissipated to the surroundings of the human body by conduction, convection, radiation, evaporation of the moisture from skin and through respiration which is a real word situation that has been formulated into mathematical models by many scholars (Sevilgen, and Kilic, 2011) [6].

The geometry of human head is very vital to the rate of metabolism that takes place in the entire body since the brain is situated at the head and it controls the entire body mechanism which leads to the heat generation. Heat transmission in human head is determined by the geometry of the human head. Our focus in this work is to look at the effects of the shape of the human head on heat transmission in the presence of heat loss for better efficiency of the body mechanism (Makinde, 2010) [3]. This is because the temperature of the human head is influenced by the thermogenesis slope parameters, thermogenesis heat production parameter and the Biot numbers of the human head.to examine the role of a more detailed thermos physical properties influence on the distribution of temperature in human head. Here, the variable heat loss is introduced to the existing model with Newtonian heating to reduce the temperature of human head for proper functioning of the entire system. Thus the study seeks to: formulate a mathematical model to represent geometry of human head; investigate the influence of heat loss with Newtonian heating on the heat distribution of human head; investigate the effects of the thermogenesis heat production parameter on the geometry of human head; predict a steady state solution to the formulated problem on the geometry of human head.

The significance of the study rests in the fact that it shall provide a better understanding on the nonlinearity of temperature distribution of human head and thereby augment its usefulness in various medical treatments of ailments that are controlled by the temperature distribution of human head with heat loss parameters in order to reduce the heat distribution in human body control by the head geometry.

Mathematical analysis of the model

Model formulation

We consider a steady state one-dimensional spherically symmetrical heat conduction equation modelling a simplified dimension equation for heat of human head geometry. Following Makinde (2010) [3], the governing energy balance equation together with its corresponding boundary conditions then becomes:

Existing Model

$$\frac{d^2T}{d\bar{r}^2} + \frac{2}{\bar{r}}\frac{dT}{d\bar{r}} + \frac{\delta}{k}e^{-\alpha T} = 0 \tag{3.1.1}$$

$$\frac{dT}{d\bar{r}} = 0, \qquad on \quad \bar{r} = 0 \tag{3.1.2}$$

$$-K\frac{dT}{d\bar{r}} = \beta(T - T_a), \qquad on \quad \bar{r} = R \tag{3.1.3}$$

From equation (3.1.1) we introduced heat loss variable to obtain the modified model:

Modified Model

Shape of human head means the geometry of human head. The human head is spherical in shape which guarantees the level of temperature distribution. As shown in the model equation.

$$\frac{1}{\bar{r}^n}\frac{d}{d\bar{r}}\left(\bar{r}^n\frac{dT}{d\bar{r}}\right) + \frac{\delta}{k}e^{-\alpha T} - \beta_1\frac{S}{V}T = 0 \tag{3.1.4}$$

Where r is the radial distance measure, n is the geometry of the vessel. When n = 0, we talk about slab, n = 1, we talk about cylindrical and when n = 2, then we talk of spherical in nature. From the equation, n = 2, therefore the human head is spherical which represents the geometry of human head.

When you differentiate the above equation (3.1.4) we have

$$\frac{d^2(T)}{dr^2} + \frac{2}{r}\frac{d(T)}{dr} + \frac{\delta}{k}e^{-\alpha T} - \beta_1\frac{S}{V}T = 0 \tag{3.1.5}$$

Where k is the thermal conductivity inside the head,

T is the absolute temperature, $0 \le \bar{r} < R$ is the radial distance measure from the centre to the periphery of head, β is a heat exchange coefficient from the head to the surrounding medium, T_a is the ambient temperature, α and δ are the metabolic thermogenesis slope parameter and thermogenesis heat production parameter respectively.

β_1 is the convection coefficient of the head medium, S is the surface area of the human head V is the volume of the human head.

Equations (3.1.2-3.1.5) are made dimensionless by introducing the following variables:

$$\theta = \frac{T}{T_a}, \; r = \frac{\bar{r}}{R}, \vec{r} = rR \tag{3.1.6}$$

Where θ is dimensionless temperature

From equation (3.1.6), we have

$$d\bar{r} = Rdr \tag{3.1.7}$$

From equations (3.1.6) and (3.1.7). Then, we have the following derivatives

$$\frac{1}{R}\frac{d}{dr} = \frac{d}{d\bar{r}}, \quad \frac{1}{R^2}\frac{d^2}{dr^2} = \frac{d^2}{d\bar{r}} \; and \; \frac{2}{\bar{r}} = \frac{2}{rR} \tag{3.1.8}$$

Using equations (3.1.6-3.1.8) in equation (3.1.5), we obtain

$$\frac{1}{R^2}\frac{d^2(\theta T_a)}{dr^2} + \frac{2}{R^2 r}\frac{d(\theta T_a)}{dr} + \frac{\delta}{k}e^{-\alpha\theta T_a} - \beta_1\frac{S}{V}\theta T_a = 0 \tag{3.1.9}$$

Multiply equation (3.1.9) by $\frac{R^2}{T_a}$, we have

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{\delta R^2}{T_a k}e^{-\alpha\theta T_a} - \frac{\beta_1 SR^2}{V}\theta = 0 \tag{3.1.10}$$

Therefore, equation (3.1.10) becomes

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \lambda e^{-m\theta} - \gamma\theta = 0 \tag{3.1.11}$$

Where $m = \alpha T_a, \; \lambda = \frac{\delta R^2}{kT_a}, \; \gamma = \frac{\beta_1 SR^2}{V}$ (3.1.12)

And using the same equations (3.1.6-3.1.8) for the boundary conditions (equations 3.1. 2-3.1.3), we have

$$\frac{1}{R}\frac{d(\theta T_a)}{dr} = 0, \; on \; r = 0,$$

Which gives

$$\frac{T_a}{R}\frac{d\theta}{dr} = 0, on \; r = 0 \tag{3.1.13}$$

Equation (3.1.13) gives

$$\frac{d\theta}{dr} = 0, \quad on \; r = 0 \tag{3.1.14}$$

Similarly, equation (3.1.3) also gives

$$-k\frac{1}{R}\frac{d(\theta T_a)}{dr} = \beta(\theta T_a - T_a),$$

Which resulted into

$$\frac{d\theta}{dr} = \frac{\beta R}{k}(1-\theta), \; on \;\; r = 1$$

This can be written as

$$\frac{d\theta}{dr} = B_i(1-\theta), \; on \;\; r = 1 \tag{3.1.15}$$

Where $B_i = \dfrac{\beta R}{k}$ (3.1.16)

Hence, the dimensionless temperature distribution equation for human head with boundary conditions can be summarized by (3.1.11), (3.1.14) and (3.1.15)

Method of Solution

Here, we discussed the classical Runge-Kutta of order 4 used in solving this model problem alongside with the shooting technique. The review of Runge-Kutta, Shooting technique and the Maple 18 package used are discussed as follows:

Runge-Kutta of Order 4

The numerical solutions for the set of transformed ordinary differential equations have been computed by applying Runge-Kutta method together with the shooting technique. This method is concisely outlined below.
Consider the ordinary differential equation

$$\frac{dy}{dx} = f(x,y) \, with \;\; y(x_n) = y_n \tag{3.4.1}$$

By Runge-Kutta method, the next value y_{n+1} is given by

$$y_{n+1} = y_n + \left(k_1 + \left(2+\sqrt{2}\right)k_2 + \left(2+\sqrt{2}\right)k_3 + k_4\right)/6 \tag{3.4.2}$$

Where

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \left(\sqrt{2}-1\right)\frac{k_1}{2} + \left(2-\sqrt{2}\right)\frac{k_2}{2}\right)$$
$$k_4 = hf\left(x_n + h, y_n - \sqrt{2}\frac{k_2}{2} + \left(1+\frac{\sqrt{2}}{2}\right)k_3\right) \tag{3.4.3}$$

Shooting Technique

One of the appropriate numerical techniques to obtain the solutions for initial value problems and boundary value problems is shooting method. Shooting technique is implemented in order to determine the missing wall conditions. In this method, an arbitrary value is assumed for the derivatives of chosen function initially. The special feature of shooting method is its approach to convert a boundary value problem into an equivalent initial value problem. The resulting initial value problem, finally, is solved by applying some numerical methods such as Runge-Kutta to obtain the required results. A brief discussion of shooting method for a boundary value problem is carried out below.
Consider the two-point boundary value problem:

$$u'' = f(t,u,u'), \;\; a < t < b \tag{3.5.1}$$

Together with boundary conditions

$$u(a)=\alpha,\ u(b)=\beta \tag{3.5.2}$$

Let us assume $y_1 = u,\ y_2 = u' \Rightarrow y_1{}' = y_2\ \ now\ \ y_2{}' = u''$ (3.5.3)

The above boundary value problem can be written as

$$\begin{pmatrix} y_1{}' \\ y_2{}' \end{pmatrix} = \begin{pmatrix} y_2 \\ f(t, y_1, y_2) \end{pmatrix},\ a<t<b \tag{3.5.4}$$

Further, the boundary condition is $y_1 = u.$
This can be written in the matrix form as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1(a) \\ y_2(a) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1(b) \\ y_2(b) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.5.5}$$

Which is an initial value problem. The solution of such problem is determined by adjusting it to the boundary conditions by initially assuming a value for a missing derivative like u' for the above problem. Thereafter, we can apply any numerical methods to solve the ordinary differential equation and it can be observed that the guessing value matches the boundary conditions at the end. This process is repeated until the boundary conditions are satisfied.
Note that Runge-Kutta is a numerical computation scheme used to handle the initial value problem while shooting techniques is used to convert the boundary value problem into an initial value problem before it can be handled with Runge-Kutta of order 4. Hence the use of these techniques is justified.

Sample Software Package
Maple 18 software package is used to solve mathematical problems such as boundary value problems for ordinary differential equations of the form $y' = f(x, y, p),\ a \le x \le b,$ by applying a collocation method subject to general non-linear, two-point boundary conditions $f(y(a), y(b), p) = 0.\ Here,\ p$ is a vector of unknown parameters. It can also solve partial differentia equations but with restrictions. It can simulate biomathematics and mathematical models. Any boundary value problems can be formulated to get solutions with less consumption of time by using the Maple software. The first step in programming with Maple is to write the ordinary differential equation of any order into a system of first order differential equations.

When $\gamma = 0$, equation (3.1.10) can be solved by Maple satisfying equations. (3.1.13 - 3.1.14) to give

$$e^{m\theta}(1-\lambda) = -\frac{\lambda}{3B_i} \tag{3.5.1}$$

Then, there are three cases of solutions

Case I when $\lambda = 0$, then

$$\theta(r) = 1 \tag{3.5.2}$$

Case II when $m = 0$

$$\theta = 1 + \frac{\lambda}{3B_i} \tag{3.5.3}$$

Note that the source of case I and II are from the assumption made from the formulated human head model by making $\gamma = 0\ and\ \lambda = 0$.The heat loss parameter is zero and we have case I and also when the thermophoresis heat production parameter is zero, we have case II.

Case III when $m \neq 0$ and $\lambda \neq 0$

$$e^{m\theta} = 1 + m\theta + \frac{(m\theta)^2}{2!} + \frac{(m\theta)^3}{3!} + \ldots\ldots$$

$e^{m\theta}(1-\theta) = \frac{-\lambda}{3B_i}$ becomes

$(1+m\theta)(1-\theta) = \frac{-\lambda}{3B_i}$ by truncating $e^{m\theta}$ at second term, then

$$\theta = \frac{(m-1) \pm \sqrt{(m-1)^2 + 4m\left(1+\frac{\lambda}{3B_i}\right)}}{2m} \tag{3.5.4}$$

Again, if $\gamma \neq 0$, then we convert the boundary value problem of equation (3.1.10) satisfying equations (3.1.13-3.1.14) to an initial value problem of the form

$$\begin{aligned} Let\ x_1 &= r \\ x_2 &= \theta \\ x_3 &= \theta' \end{aligned},$$

Then we have

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \theta' = x_3 \\ \theta'' = \gamma x_2 - \lambda e^{-mx_2} - \frac{2x_3}{x_1} \end{pmatrix}$$

Which gives

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \gamma x_2 - \lambda e^{-mx_2} - \frac{2x_3}{x_1} \end{pmatrix} \tag{3.5.5}$$

Satisfying the initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_2 \\ 0 \end{pmatrix} \tag{3.5.6}$$

Existence and uniqueness of solution

In this section, we formulated theorem on the existence and uniqueness of solution and establish the proof of the theorem.

$(H1): If\ \beta_2 > 0,\ 0 \le x_1 \le 1,\ \ 0 \le x_2 \le l,\ \&\ a \le x_3 \le b,$ Where l, a, b are positive constants

Theorem
If $(H1)$ holds, then equation (3.5.5) has a unique solution satisfying (3.5.6).

Proof
We let

$$z' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}, \tag{3.5.7}$$

f_1, f_2, f_3 are given by equation (3.5.5). Clearly, $\frac{\partial f_i}{\partial x_j}$ is bounded for $i, j = 1, 2, 3.$ Thus,

$f_i, i = 1, 2, 3,$ are Lipchitz continuous. Hence there exists a unique solution of equation (3.5.5) satisfying equation (3.5.6) for equations (3.1.10) and (3.1.13).

Results
In this section, we display the results of the solved problem in equation (3.5.2) when $\lambda = 0$

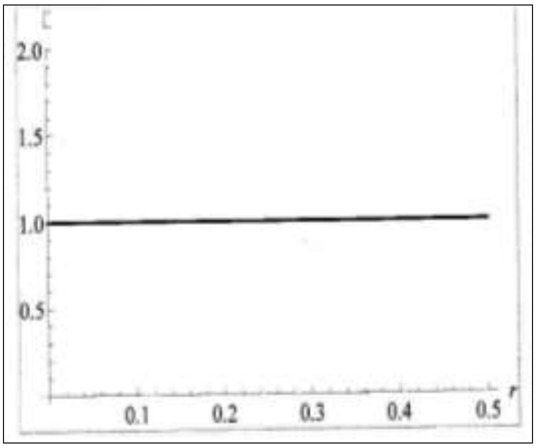

Fig 1: Graph of temperature distribution θ against r when $\lambda = 0, then\ \theta(0) = 1$.

The graphical solution to equation (3.5.3) when m = 0 is plotted below

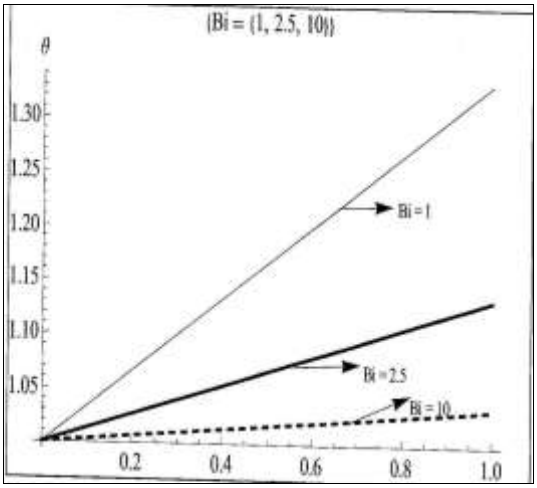

Fig 2: The temperature distribution of θ against λ for different B_i

The graphical solution to equation (3.5.3.)

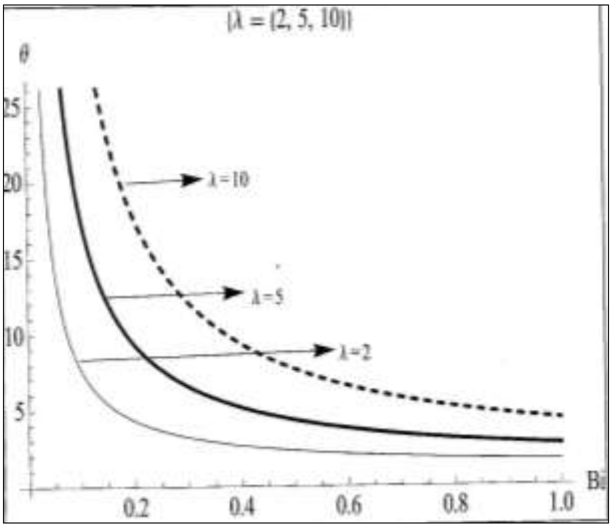

Fig 3: The temperature distribution of θ against B_i, for different λ

We then obtain the following graphs from equation (3.5.4) for Figure 4 to 7.

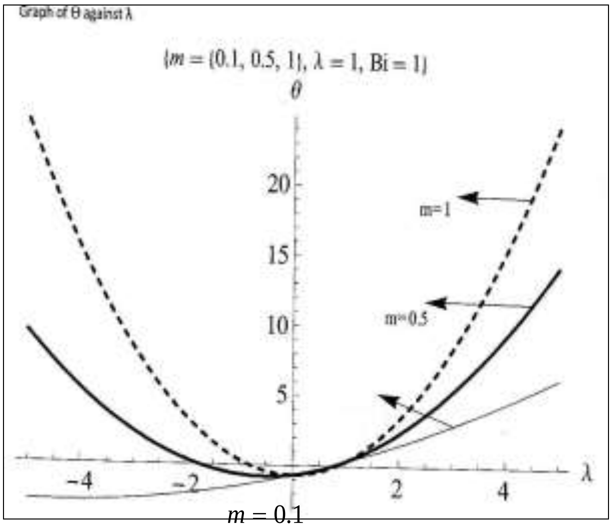

Fig 4: Temperature distribution of θ against λ; $B_i = 1$ for different m

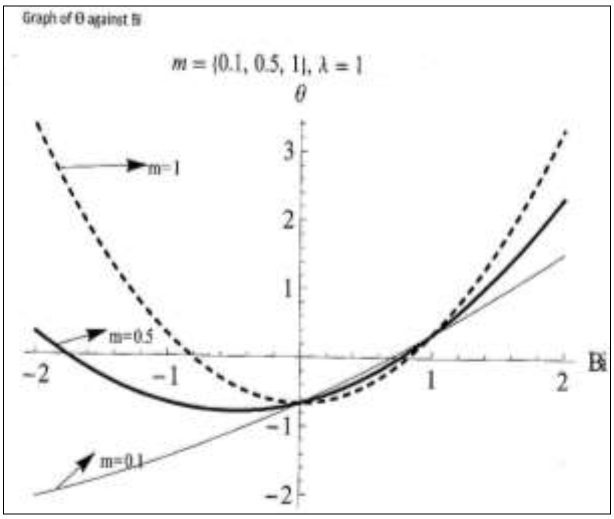

Fig 5: Temperature distribution of θ against B_i; $\lambda = 1$ for different m

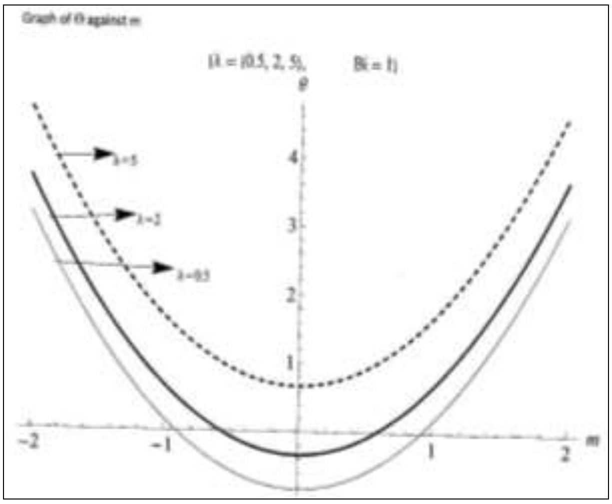

Fig 6: Temperature distribution of θ against m, $B_i = 1$ for different λ

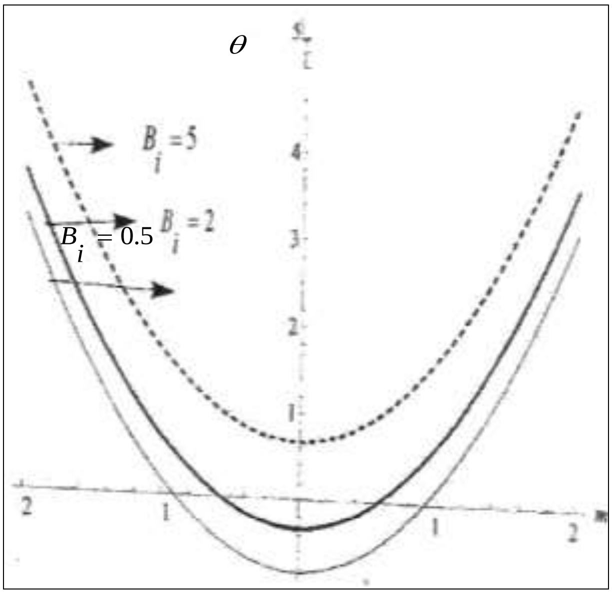

Fig 7: Temperature distribution of θ against m, $\lambda = 1$ for different B_i

Table 1: Comparison showing the core temperature when $\gamma = 0$ with Makinde (2010)

Bi	λ	**M**	**Makinde (2010)** $\theta(0)$	**Present result** $\theta(0)$
0.1	1.0	1.0	1.670707772962304	1.670707772962304
0.5	1.0	1.0	1.246067802886287	1.246067802886287
1.0	1.0	1.0	1.160819819590054	1.160819819590054
1.0	3.0	1.0	1.395994649701994	1.395994649701994
1.0	5.0	1.0	1.569878890257090	1.569878890257090
1.0	1.0	0.5	1.270842849149902	1.270842849149902
1.0	1.0	0.1	1.436256582344328	1.436256582344328

Table 2: Computations showing the core temperature at various parameter values in the human head model.

λ	γ	m	Bi	$\theta(0)$
0.1	0.1	1	0.1	0.850016858237469863
0.5	0.1	1	0.1	1.152840655938161880
1.0	0.1	1	0.1	1.393793171986422260
0.1	0.5	1	0.1	0.435081496158957748
0.1	1.0	1	0.1	0.265796930073086580
0.1	0.1	3	0.1	0.764695401559435162
0.1	0.1	5	0.1	0.744946172574163624
0.1	0.1	1	10	0.987598639552977554

0.1	0.1	1	100	0.989426551103024910
0.1	0.1	1	500	0.989589753578274722
0.1	0.1	1	1000	0.989610162204493382

We then obtain the following graphs from equations (3.5.5) and (3.5.6) for figure 8 to 11.

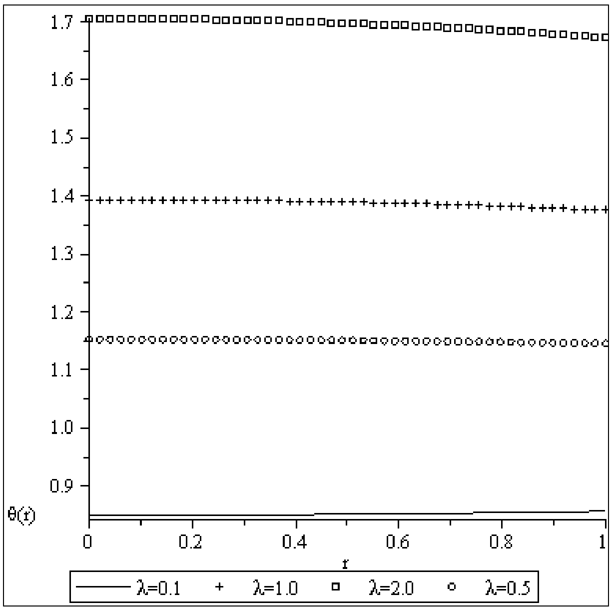

Fig 8: Effects of thermogenesis heat production parameter λ for fixed values of m = 1, Bi = 0.1, $\gamma=0.1$.

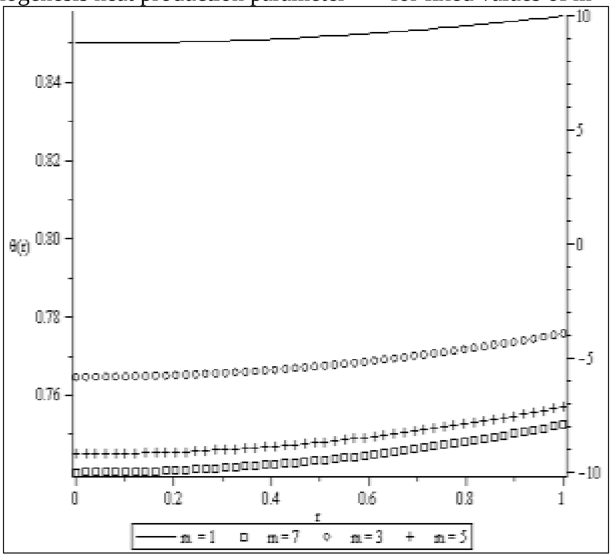

Fig 9: Effects of metabolic thermogenesis slope parameter m for fixed values of Bi = 0.1, $\gamma=0.1$, $\lambda=0.1$

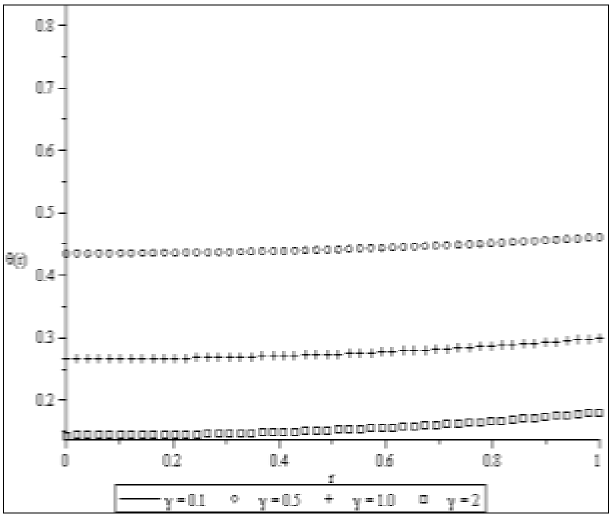

Fig 10: Effects of variable heat loss parameter γ for fixed vales of Bi = 0.1, $m=1$, $\lambda=0.1$.

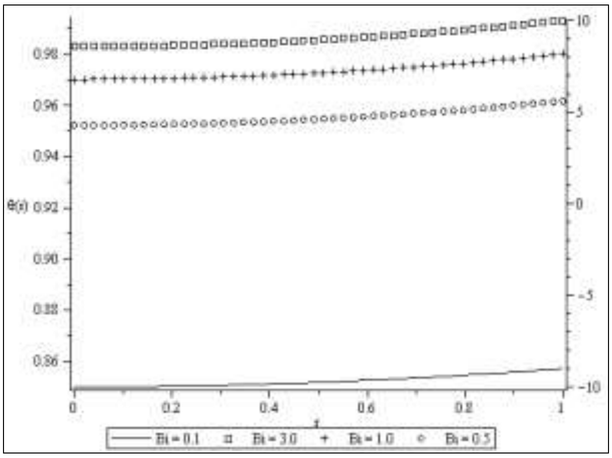

Fig 11: Effects of Biot number Bi for fixed values of $\lambda=0.1$, $m=1$, $\lambda=0.1$.

Discussion of Results

Considering the graphs in cases I, II, and III above, figure1 shows that when the thermogenesis heat production parameter $\lambda = 0$, then the temperature of the human head is constant and at a set point. Also, figure 2 and figure 3 show that the temperature of the human head increases or decreases as the Biot number decreases or increases and thermogenesis heat parameter increases or decreases. This is in agreement with the earlier results obtained by Celik and Gokman (2003) [1] together with Makinde (2010) [3].

From figure 4, and 5, an increase in the thermogenesis heat parameter or increase in metabolic thermogenesis slope parameters may lead to increase in the temperature of the human head. This is in agreement with Makinde (2010) [3]. In addition to this, figure 6 and 7 show that the temperature of the human head is higher at the core and decreases transversely with minimum value at the periphery. The minimum values increase with increase in thermogenesis heat or slope parameters and increase in Biot numbers.

We present both the numerical and graphical results for the human head temperature distribution based on the numerical method. Table 1 shows the perfect agreements with Makinde (2010) [3] on the core temperature of human head. Table 2 illustrates the influences of the embedded parameters in the human head model. When λ increases, the core temperature increases. Similarly, when the heat loss parameter increases, the core temperature of human head decreases, while the biot number enhances the core temperature of human head. Figures 8 - 11 illustrate the temperature profiles in a human head as demonstrated by the computational result in Table 1. Generally, it is very interesting to note that human head temperature is higher at the core and decreases transversely with minimum value at the periphery. In Figure 8, we observed that an increase in the rate of thermogenesis heat production in a human head resulting from cellular metabolism (as normally experienced during feverish condition) may lead to an elevation in human head temperature; however, an increase in the magnitude of metabolic thermogenesis slope parameter may cause a slight decrease in human head temperature as demonstrated in Figure 9. Figure 10 depicts the temperature distribution of human head against r with various values of the heat loss parameter. It was established that as γ increases, the temperature thickness decreases across the geometry of human head. An increase in the Biot number due to an increase in the rate of heat-transfer at the interface of human head with the surrounding environment causes a general cooling effect and a reduction in human head temperature as shown in Figure 11. This is in agreement with the earlier results obtained by Çelik and Gokmen (2003) [1].

Conclusion

This study discussed graphically the solution of the human head energy balance equation with heat loss and Newtonian heating. The process reveals accurately the human head profile and confirms the earlier results reported in the literature (Makinde (2010) [3]. In this research work, shooting technique alongside with classical Runge-Kutta method is employed to obtain a solution of a nonlinear model of heat conduction in the human head. The procedure reveals accurately the human head temperature profiles and confirms earlier results reported in the literature. The distribution of the temperature in the human head is very sensitive to environment temperature changes at the periphery due to convective heat exchange with the ambient air. Also, it can be said that the temperature at the center of the head is not affected by the environmental temperature as well. Similarly, the heat loss parameter has contributed greatly to the reduction in human head temperature distribution across the head geometry.

Recommendations

This research work of a nonlinear heat conduction model of the human head can be solved with different numerical methods and the rate of convergence of the solutions obtained compared, this could be another interesting topic for further research.

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