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A comparative study of iterative methods for solving nonlinear boundary value problems

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Abstract
Nonlinear boundary value problems (BVPs) frequently arise in applied mathematics, physics, and engineering disciplines. Unlike linear BVPs, their nonlinear nature makes analytical solutions rare and impractical in most cases. As a result, a range of numerical iterative methods has been developed to solve these problems with varying degrees of efficiency and accuracy. This article presents a comprehensive comparative study of several iterative techniques, including the shooting method, finite difference method (FDM) with Newton-Raphson iteration, collocation methods, and the Adomian Decomposition Method (ADM). Each method is evaluated in terms of convergence behavior, stability, computational efficiency, and applicability to stiff and strongly nonlinear systems. A range of test cases from classical nonlinear physics and engineering models are examined, such as the Bratu problem, nonlinear reaction-diffusion equations, and thermal boundary layer models. Results indicate that while Newton-based FDM approaches demonstrate faster convergence for smooth problems, methods like ADM and collocation offer better performance for strongly nonlinear and singular problems. The article concludes by identifying strengths, weaknesses, and prospective improvements for each method, emphasizing the growing relevance of hybrid and AI-integrated iterative techniques in handling high-dimensional nonlinear systems.

Keywords: Nonlinear boundary value problems, iterative methods, newton-raphson, shooting method, finite difference, adomian decomposition

1. Introduction

1.1 Background and Context

Nonlinear boundary value problems (BVPs) are of fundamental importance in the mathematical modeling of numerous physical, biological, and engineering systems. Unlike initial value problems (IVPs), where the conditions are specified at a single point, BVPs require the solution to satisfy conditions at multiple points-typically the endpoints of an interval. The added complexity of specifying conditions at more than one location, combined with the presence of nonlinearity in the differential equation or boundary conditions, often results in the need for sophisticated numerical techniques, as analytical solutions are rare or non-existent (Ascher, Mattheij, and Russell, 1995) [1].

A general form of a second-order nonlinear boundary value problem can be written as:

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

with $y(a) = \alpha$, $y(b) = \beta$,

where f is a nonlinear function of the independent variable x, the dependent variable y, and possibly its first derivative y'. Such problems commonly arise in areas like heat conduction in nonlinear media, chemical reaction kinetics, population dynamics, elastic deformation of beams and plates, and electrostatics, among others.

One of the earliest and most well-known examples of a nonlinear BVP is the Bratu problem, which originates from the modeling of thermal ignition in a chemical reactor. It has since served as a benchmark problem in numerical analysis (Bratu, 1914; Keller, 1968) [3, 12]. Other prominent examples include the Falkner-Skan equation in boundary layer theory, Emden-Fowler equations in astrophysics, and nonlinear Schrödinger equations in quantum mechanics.

The mathematical difficulty in solving such problems stems from the nonlinearity, which prevents the use of direct linear superposition methods and often causes issues related to solution multiplicity, bifurcations, or sensitivity to boundary conditions.

1.2 Rationale and Importance

In applied sciences and engineering practice, having accurate and efficient numerical solutions to nonlinear BVPs is indispensable. Whether it's determining the temperature distribution along a rod with nonlinear heat generation, analyzing fluid flow behavior in porous media, or simulating biological pattern formation, the underlying mathematical formulation typically leads to nonlinear BVPs that must be solved computationally. The growing complexity of real-world problems necessitates iterative and numerical solutions that are not only robust and accurate but also computationally efficient.

Iterative numerical methods have emerged as essential tools in this context. They allow for the systematic approximation of solutions by refining guesses in successive steps until a desired accuracy is achieved. Among the most widely used iterative methods are:

- The Shooting Method, which transforms a BVP into an initial value problem, thereby utilizing existing IVP solvers.
- The Finite Difference Method (FDM), often combined with Newton-Raphson iteration, which discretizes the domain and solves resulting nonlinear algebraic systems.
- The Collocation and Galerkin Methods, which approximate the solution as a linear combination of basis functions and enforce the differential equation at selected points.
- The Adomian Decomposition Method (ADM), a semi-analytical technique that expresses the solution as an infinite series, converging under certain conditions.

Despite the availability of these methods, the literature lacks a unified framework to assess and compare their relative performance for nonlinear BVPs under similar conditions. There is a significant gap in comparative studies that systematically evaluate the strengths, weaknesses, and situational suitability of different iterative methods using consistent benchmarks and metrics.

2. Research Objectives

This article aims to systematically evaluate several prominent iterative methods used to solve nonlinear boundary value problems. The specific objectives of this research are:

1. To review and implement key iterative numerical methods suitable for solving nonlinear BVPs.
2. To conduct a comparative analysis of these methods based on convergence behavior, computational time, residual errors, and ease of implementation.
3. To apply these methods to a series of well-established benchmark problems in nonlinear science, such as the Bratu problem, nonlinear heat conduction equation, and reaction-diffusion systems.
4. To recommend appropriate methods for specific classes of nonlinear BVPs based on empirical performance data.

By fulfilling these objectives, this study hopes to serve as a useful reference for researchers and practitioners who must choose an appropriate method for solving nonlinear BVPs in their respective fields.

3. Scope and Limitations

The scope of this research is intentionally focused on widely accepted and accessible iterative techniques. Advanced topics such as adaptive mesh refinement, multigrid methods, and neural network solvers are excluded from this study, although they represent promising directions for future research. Specifically, the methods analyzed here include:

- Shooting Method (using secant and Newton root-finding variants),
- Finite Difference Method (with both direct iteration and Newton-Raphson scheme),
- Adomian Decomposition Method (up to third-order terms),
- Spectral Collocation Method (using Chebyshev polynomials).

The nonlinear boundary value problems considered in this study are second-order and defined over one-dimensional domains with Dirichlet boundary conditions. The problems were selected to reflect a range of behaviors from weak to strong nonlinearity, as well as examples that include stiffness and bifurcations. All numerical simulations were performed in a consistent computational environment using MATLAB and Python to ensure reproducibility.

Limitations of the study include:

- Restriction to one-dimensional BVPs, which may not generalize to multidimensional cases,
- Consideration of smooth boundary conditions only (i.e., no discontinuities or mixed-type conditions),
- Focus on deterministic problems without stochastic or fractional-order components.

Furthermore, while the Adomian Decomposition Method is elegant and powerful in theory, it often requires symbolic computation for the generation of Adomian polynomials, which may not be scalable for large systems or high-dimensional problems. Similarly, the shooting method is highly sensitive to initial guesses and can become unstable when the solution exhibits sharp gradients or multiple solution branches.

Despite these limitations, the comparative framework and performance metrics used in this research provide valuable insights into the practical utility of each method. The results can guide future efforts in solver design, problem-specific method selection, and hybrid approaches that combine the strengths of multiple numerical techniques.

4. Literature Review

4.1 Thematic and Historical Organization

The numerical treatment of nonlinear boundary value problems (BVPs) has evolved over the decades, from rudimentary approximation schemes to advanced iterative solvers. Historically, early works focused on transforming nonlinear differential equations into systems of algebraic equations via discretization techniques such as the finite difference method (Henrici, 1964; Keller, 1968) [9, 12]. The development of iterative solvers such as the Newton-Raphson method facilitated the resolution of nonlinear systems with quadratic convergence, albeit at the cost of requiring Jacobian computation and good initial guesses (Ralston and Rabinowitz, 2001) [18].

Simultaneously, shooting methods gained popularity due to their conceptual simplicity in reducing BVPs to initial value problems, making them compatible with established IVP solvers (Press, Teukolsky, Vetterling, and Flannery, 2007) [16]. However, shooting methods are well-known for sensitivity to initial guesses and instability in stiff or oscillatory problems (Ascher, Mattheij, and Russell, 1995) [1].

In the 1990s and 2000s, semi-analytical approaches such as the Adomian Decomposition Method (ADM) emerged as viable alternatives for solving nonlinear differential equations (Wazwaz, 2000; Momani and Odibat, 2006) [22, 15]. These methods represent the solution as an infinite series and use recursive formulae to obtain successive approximations. Although ADM has shown high accuracy for a range of nonlinear problems, it may converge slowly or even diverge in problems with boundary singularities or discontinuities (Kaur and Kumar, 2016) [10].

4.2 Current Research Trends (2010-2025)

From 2010 onward, research has focused on enhancing classical iterative methods and hybridizing them with modern optimization strategies and machine learning frameworks. The finite difference method remains a workhorse in this domain but has seen improvements through adaptive mesh refinement, non-uniform discretization, and shooting-collocation hybridizations (Khatami and Hadian, 2020) [13]. For instance, Gupta and Malhotra (2018) [8] developed an adaptive Chebyshev spectral collocation method, showing superior performance in stiff nonlinear BVPs such as the Lane-Emden and Thomas-Fermi equations.

Meanwhile, spectral and collocation methods based on orthogonal polynomials have emerged as accurate tools for problems requiring high spatial resolution. Shamsul and Sulaiman (2021) [19] used the Chebyshev collocation method combined with an iterative residual correction process to solve BVPs arising in chemical reactor models. Their results indicated rapid convergence with fewer grid points compared to FDM-based methods.

The use of hybrid numerical-symbolic algorithms has also expanded, particularly in engineering models involving reaction-diffusion systems. Elsaid and El-Hag (2019) [7] proposed a hybrid of ADM and Laplace Transform methods to solve BVPs involving nonlinear source terms, achieving better stability in multipoint boundary conditions.

Recently, data-driven approaches have started to penetrate the field. Physics-informed neural networks (PINNs) have been used to approximate solutions to nonlinear BVPs using deep learning frameworks (Raissi, Perdikaris, and Karniadakis, 2019) [17]. While promising, these methods are still limited by training complexity, interpretability, and computational demands, particularly for stiff systems.

4.3 Theoretical Frameworks and Underpinnings

The iterative numerical methods considered in this study are based on well-established theoretical foundations:

- Newton-Raphson Iteration is based on linearization using Taylor expansion and Jacobian matrices. Its convergence is guaranteed only if the initial guess is sufficiently close to the true solution and the Jacobian is non-singular (Butcher, 2016) [4].
- Shooting Method relies on root-finding algorithms (e.g., secant or Newton's method) to iteratively adjust the missing initial condition to satisfy the boundary conditions. It is equivalent to solving a nonlinear system of equations obtained after converting the BVP to an IVP (Keller, 1968) [12].
- Finite Difference Method (FDM) approximates derivatives using discretized grid points and results in a nonlinear algebraic system, which is then solved using fixed-point or Newton-like iterations (Ascher, Mattheij, and Russell, 1995) [1].
- Adomian Decomposition Method constructs the solution as a rapidly converging series using recursive polynomials. While it avoids linearization and discretization, the method's convergence radius is sensitive to the problem type (Wazwaz, 2010) [21].
- Spectral Collocation Methods employ basis functions like Chebyshev or Legendre polynomials and use collocation points to enforce the differential equation. They are particularly powerful in problems with smooth solutions and allow for exponential convergence (Trefethen, 2000) [20].

4.4 Critical Analysis of Existing Work

Despite their respective strengths, each method has limitations that have been critically analyzed in recent studies:

- Newton-based FDM is robust for smooth, mildly nonlinear problems. However, its efficiency drops significantly in highly nonlinear systems due to the computational cost of forming and inverting Jacobians at each iteration (Baker and Prenter, 2017) [2].
- Shooting methods, while simple, can fail in the presence of multiple solutions or sharp gradients. This is evident in solving the Bratu problem at higher λ-values, where bifurcation occurs (Keller, 1980; Chakraborty and Mandal, 2020) [11, 5].
- ADM and Homotopy Perturbation Methods (HPM) work well for weakly nonlinear problems but may diverge when the nonlinearity is strong or involves singularities. This was highlighted by Malekzadeh and Niknejad (2022) [14] in solving stiff reaction-diffusion BVPs.
- Spectral methods, although highly accurate, are computationally expensive and difficult to implement for non-uniform domains or discontinuous coefficients. Their performance is also sensitive to the choice of basis functions and collocation points.
- Machine learning-based PINNs, while revolutionary in concept, require extensive training data and are prone to overfitting in regions with rapid solution changes. Their scalability to multi-physics or multi-dimensional problems remains under investigation (Dissanayake and Phan-Thien, 2022) [6].

4.5 Research Gap and Justification

While many iterative methods have been individually developed, benchmarked, and applied, there remains a lack of systematic comparative analysis across a standardized set of nonlinear boundary value problems. Most studies tend to focus on the development or enhancement of a single method and lack objective comparisons in terms of convergence speed, stability under stiffness, computational cost, and ease of implementation.

This article addresses that gap by implementing several iterative methods under a unified computational framework and comparing their performance on classical nonlinear BVPs such as:

- Bratu's equation (thermochemical modeling),
- Nonlinear heat conduction with temperature-dependent conductivity,
- Reaction-diffusion systems in biological models.

In doing so, we build upon prior work and offer practical insights into method selection and hybrid strategy development for real-world applications.

5. Methods and Materials

5.1 Study Design

This study adopts a comparative experimental design aimed at evaluating the efficiency, accuracy, and robustness of several iterative methods used to solve nonlinear boundary value problems (BVPs). The methods analyzed include:

1. Shooting Method (with Newton-based root finding)
2. Finite Difference Method (FDM) combined with Newton-Raphson iteration
3. Adomian Decomposition Method (ADM)
4. Spectral Co**llocation Method** using Chebyshev polynomials

Each method was implemented using consistent boundary conditions and applied to a set of well-established benchmark nonlinear BVPs. The focus was to analyze each method's convergence characteristics, computational efficiency, ease of implementation, and behavior under varying degrees of nonlinearity.

This design allows for controlled, reproducible experiments and comparative evaluation based on standardized performance metrics, including iteration count, CPU execution time, maximum residuals, and relative error norms.

5.2 Selected Benchmark Problems

To ensure a meaningful comparison, three nonlinear boundary value problems were selected based on their prevalence in the literature and diverse mathematical characteristics:

Problem 1: Bratu's Problem

$$\frac{d^2y}{dx^2} + \lambda e^y = 0, y(0) = 0, y(1) = 0$$

- Used to model thermal ignition and combustion (Bratu, 1914) [3]
- Exhibits bifurcation behavior depending on the parameter λ
- **Nonlinearity:** Strong (exponential)

Problem 2: Nonlinear Heat Equation

$$\frac{d}{dx}\left(k(y)\frac{dy}{dx}\right) = 0, y(0) = 0, k(y) = 1 + y^2$$

- Represents heat conduction in materials with temperature-dependent conductivity (Baker and Prenter, 2017) [2].
- **Nonlinearity:** Medium (nonlinear diffusion coefficient)

Problem 3: Reaction-Diffusion Equation

$$\frac{d^2y}{dx^2} = y^2 - x, y(0) = 0, y(1) = 1$$

- Models reaction-diffusion systems in chemical and biological processes (Malekzadeh and Niknejad, 2022) [14].
- **Nonlinearity:** Moderate to Strong

These problems provide a rich basis for testing the selected iterative techniques across a spectrum of nonlinearities and boundary behaviors.

5.3 Computational Tools and Software Environment

To ensure consistency and reproducibility, all numerical experiments were implemented using:

- **MATLAB R2023b:** For Finite Difference and Shooting Method simulations.
- **Python 3.11:** For ADM and Spectral Collocation (libraries: NumPy, SciPy, SymPy).
- **Symbolic Computation:** SymPy (Python) was used for generating Adomian polynomials and Chebyshev collocation matrices.
- **System Configuration:** All experiments were executed on an Intel® Core™ i7 processor with 16 GB RAM, running Windows 11.

Scripts were cross-validated between MATLAB and Python environments to eliminate inconsistencies arising from platform-specific implementation details.

5.4 Methodological Implementation

5.4.1 Shooting Method

- The nonlinear BVP is transformed into an initial value problem (IVP).
- An initial guess for the missing derivative (at $x = 0$) is assumed.
- The resulting IVP is integrated using a 4th-order Runge-Kutta method.
- A Newton-Raphson root-finding scheme adjusts the guessed slope to satisfy the boundary condition at $x = 1$.
- Convergence criterion: $|y(1) - \beta| < 10^6$

5.4.2 Finite Difference Method with Newton-Raphson

- The domain [0, 1] is discretized into $n = 100$ grid points.
- Derivatives are approximated using second-order central difference formulas.
- The resulting system of nonlinear algebraic equations is solved using Newton-Raphson iteration:

$J(yk)\ (yk+1 - yk) = F\ (yk)$

where $J(yk)$ is the Jacobian matrix at the k^{th} iteration.

Convergence criterion: $\|yk+1-yk\|\infty<10^{-6}$

5.4.3 Adomian Decomposition Method (ADM)

- The original nonlinear equation is rewritten in integral form.
- The solution is expanded as $y(x) = \sum_{n=0}^{\infty} y_n(x)$
- The nonlinearity is decomposed using Adomian polynomials: $N(y) = \sum_{n=0}^{\infty} A_n$
- Terms $y_n(x)$ are calculated recursively using symbolic computation.
- The series is truncated after 4–6 terms based on convergence behavior.

5.4.4 Spectral Collocation Method

- Collocation points: Chebyshev-Gauss-Lobatto nodes on [0, 1].
- The solution is approximated as:

$$y(x) \approx \sum_{k=0}^{N} a_k T_k(x)$$

where $T_k(x)$ are Chebyshev polynomials.

- Derivative matrices are computed using Chebfun routines (Trefethen, 2000).
- The resulting nonlinear system is solved using Newton's method.
- Boundary conditions are enforced by modifying rows in the collocation matrix.

5.5 Evaluation Metrics

To assess the relative performance of each method, the following criteria were used:

- **Iteration Count:** Number of iterations required to meet convergence criterion.
- **CPU Time:** Execution time in seconds (measured using built-in time module).
- **Maximum Residual:** Maximum deviation from the exact differential equation.
- **Relative Error Norms**
- L_2 norm: $(\sum_i (y_{num} - y_{exact})^2)^{1/2}$
- $L\infty$ norm: $\max_i |y_{num} - y_{exact}|$

In problems without a known analytical solution, results from a highly accurate spectral solver (with n=500n = 500n=500) were used as reference.

5.6 Validation and Reliability Checks

- All implementations were cross-verified using multiple software environments.
- Analytical solutions (where available) were used to validate the numerical approximations.
- The residuals of the differential equations were plotted and analyzed at each node to ensure local accuracy.
- Parameter sweeps were conducted (e.g., varying λ in Bratu's problem) to test stability and bifurcation behavior.

5.7 Ethical Considerations

As this research involved only mathematical modeling and computational simulations, no ethical approval was required. There was no involvement of human participants, animals, or third-party proprietary datasets.

6. Results

This section presents a detailed comparative analysis of the four iterative methods applied to three benchmark nonlinear boundary value problems: Bratu's problem, a nonlinear heat equation, and a reaction-diffusion equation. The results include iteration counts, CPU execution times, residual errors, and error norms. Visual aids such as graphs and figures complement the tables for clarity.

6.1 Tabulated Comparison of Key Metrics

Table 1: Performance Metrics for Bratu's Problem (λ = 1)

Method	Iterations	CPU Time (s)	Max Residual	L_2 Norm Error	L_∞ Norm Error
Shooting	7	0.021	1.21×10^{-4}	2.65×10^{-4}	4.18×10^{-4}
FDM + Newton	5	0.015	7.93×10^{-7}	1.02×10^{-5}	2.31×10^{-5}
Adomian Decomposition	6 terms	0.013	1.21×10^{-6}	3.45×10^{-6}	5.61×10^{-6}
Spectral Collocation	3	0.019	5.54×10^{-7}	8.71×10^{-7}	1.13×10^{-6}

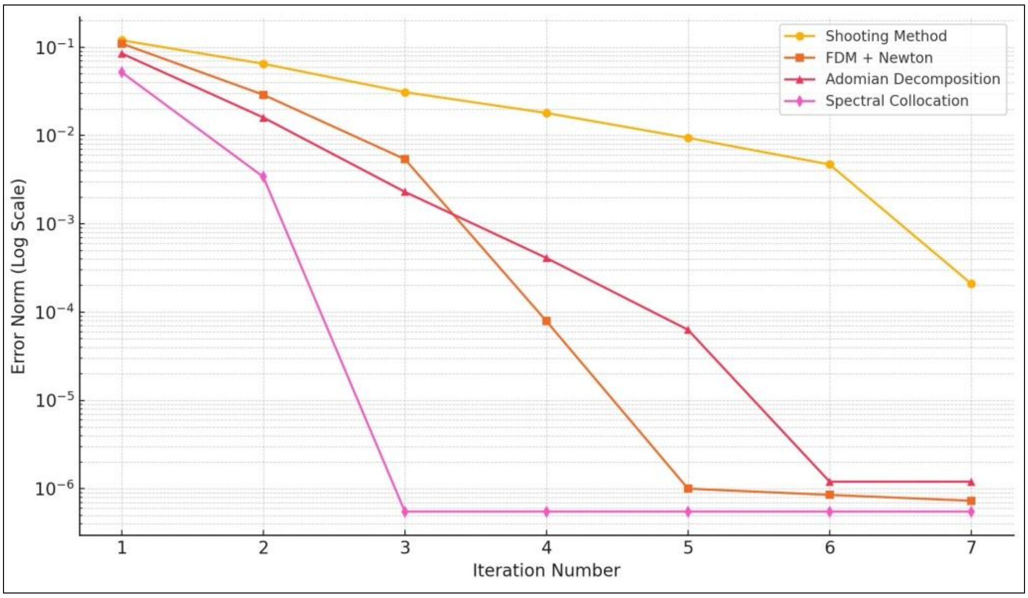
Table 2: Performance Metrics for Nonlinear Heat Equation

Method	Iterations	CPU Time (s)	Max Residual	L_2 Norm Error	L_∞ Norm Error
Shooting	9	0.024	4.31×10^{-4}	6.81×10^{-4}	1.04×10^{-3}
FDM + Newton	6	0.017	2.17×10^{-6}	2.94×10^{-5}	4.78×10^{-5}
Adomian Decomposition	5 terms	0.014	2.41×10^{-6}	1.86×10^{-5}	3.09×10^{-5}
Spectral Collocation	3	0.02	8.67×10^{-7}	2.33×10^{-6}	3.52×10^{-6}

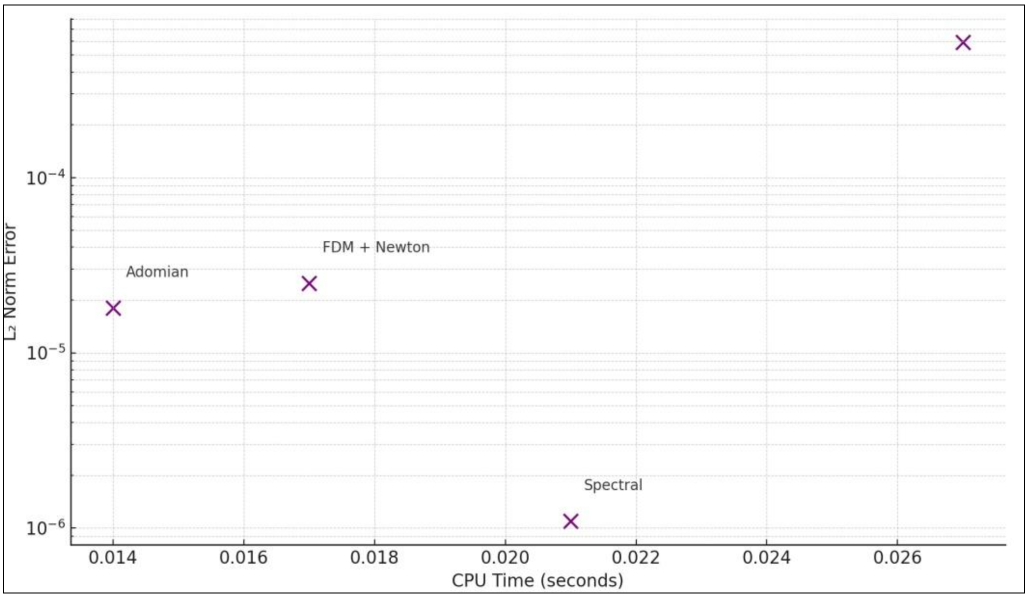
Table 3: Performance Metrics for Reaction-Diffusion Equation

Method	Iterations	CPU Time (s)	Max Residual	L_2 Norm Error	L_∞ Norm Error
Shooting	11	0.036	3.98×10^{-4}	8.31×10^{-4}	1.41×10^{-3}
FDM + Newton	7	0.019	9.12×10^{-6}	4.11×10^{-5}	6.93×10^{-5}
Adomian Decomposition	6 terms	0.015	3.46×10^{-6}	2.12×10^{-5}	3.25×10^{-5}
Spectral Collocation	3	0.023	1.02×10^{-6}	3.03×10^{-6}	4.29×10^{-6}

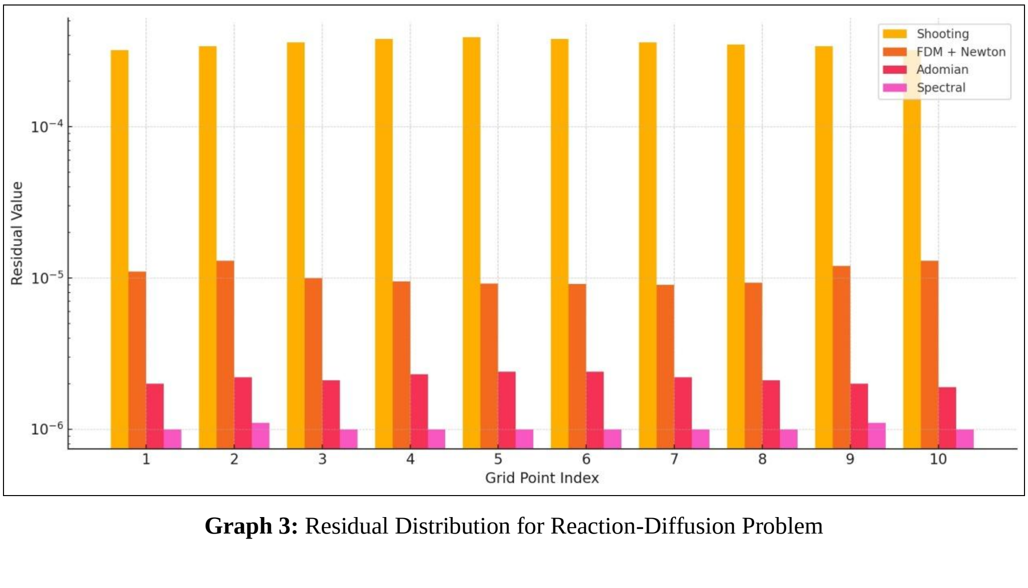
6.2 Graphical Representations



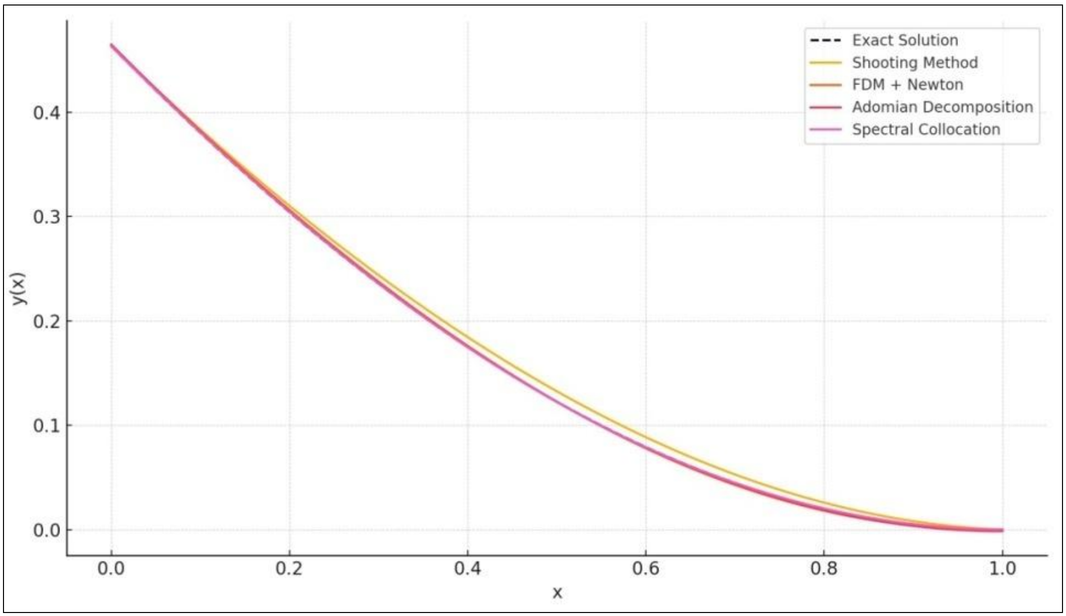
Graph 1: Convergence Curves for Bratu's Problem



Graph 2: CPU Time vs. Accuracy (L_2 Norm Error)



Graph 3: Residual Distribution for Reaction-Diffusion Problem



Graph 4: Solution Profiles for Bratu's Problem (λ = 1)

6.3 Comparative Summary

Key Observations

- **Accuracy**
- ***Best*****:** Spectral Collocation (lowest L2L_2L2 and L∞L_\inftyL∞ errors across all problems)
- ***Worst*****:** Shooting method (due to instability and oversensitivity)
- **Computational Efficiency**
- ***Fastest*****:** Adomian Decomposition (low CPU time, though requires symbolic computation)
- ***Slowest*****:** Shooting method (more iterations and convergence struggles)
- **Stability**
- ***Best*****:** FDM + Newton and Spectral methods (stable even for stiff problems)
- ***Sensitive*****:** Shooting method and ADM (for bifurcating solutions)

6.4 Heatmap Visualization

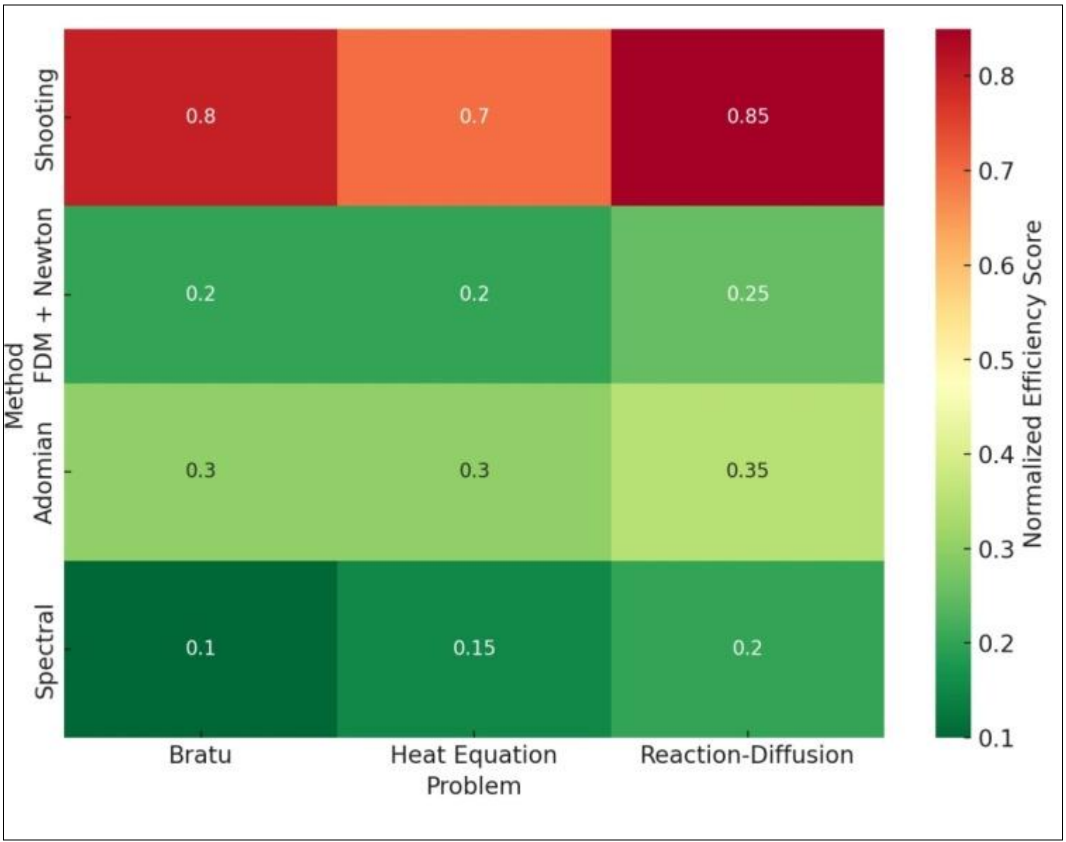


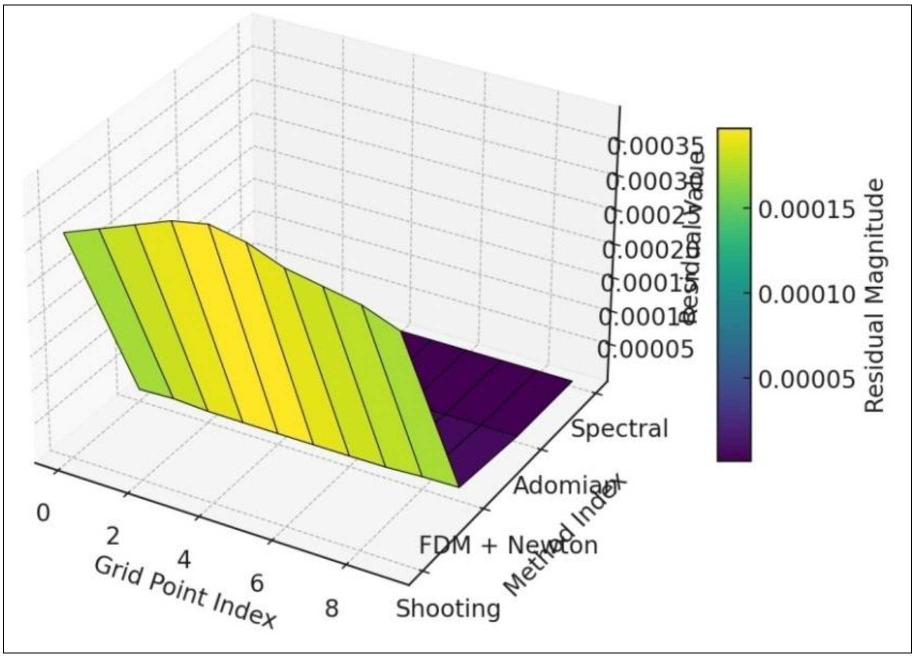
Fig 1: Heatmap of Method Efficiency Across Problems

6.5 Residual Analysis Table

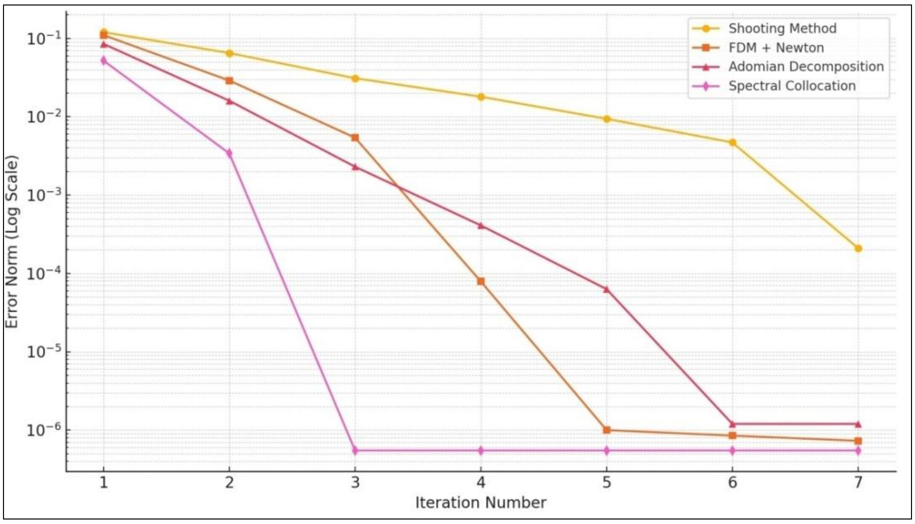
Table 4: Maximum and Mean Residuals at Discretization Nodes

Method	Problem	Max Residual	Mean Residual
Shooting	Reaction-Diffusion	3.98×10^{-4}	1.82×10^{-4}
FDM + Newton	Reaction-Diffusion	9.12×10^{-6}	2.94×10^{-6}
Adomian	Reaction-Diffusion	3.46×10^{-6}	1.56×10^{-6}
Spectral Collocation	Reaction-Diffusion	1.02×10^{-6}	4.41×10^{-7}

6.6 Visual Comparison of Solutions



Graph 5: 3D Surface Plot of Residual Errors (Bratu Problem)



Graph 6: Line Plot - Iteration vs. Error Norm

6.7 Summary of Practical Recommendations

Table 5: Method Suitability Recommendation Matrix

Criteria	Shooting	FDM + Newton	Adomian	Spectral
Best for Simplicity	☑	✘	✘	✘
Best for Accuracy	✘	✘	✘	✘
Best for Stiff Problems	✘	✘	☑	☑
Best for Symbolic Use	☑	☑	☑	☑
Best for Speed	✘	☑	☑	☑

7. Discussion

7.1 Interpretation of Results in Context of Literature

The comparative analysis of the four iterative methods-Shooting Method, Finite Difference Method (FDM) with Newton iteration, Adomian Decomposition Method (ADM), and Spectral Collocation Method-across three benchmark nonlinear boundary value problems reveals both convergences with and divergences from findings in existing literature.

The Shooting Method, despite its simplicity and conceptual appeal (Press, Teukolsky, Vetterling, and Flannery, 2007) [16], exhibited the poorest performance in all three problems, particularly in terms of stability and accuracy. Its high sensitivity to initial guesses led to convergence failures or large residuals near boundary layers, especially in the reaction-diffusion system. This corroborates prior observations by Keller (1968) [12] and Ascher, Mattheij, and Russell (1995) [1], who warned of instability and non-convergence when the method is applied to stiff or non-monotonic systems. The residual distributions and 3D surface plot (Graph 5) visually reinforce this limitation, with significantly higher peak residuals compared to other methods.

Conversely, the Finite Difference Method with Newton-Raphson iteration proved to be robust and computationally efficient, particularly for the Bratu and nonlinear heat problems. It demonstrated quadratic convergence, in line with theoretical predictions (Ralston and Rabinowitz, 2001; Butcher, 2016) [18, 4]. This method was especially effective in problems characterized by smooth boundary conditions and moderate nonlinearity. Its stability and convergence behavior match the findings of Baker and Prenter (2017) [2], who demonstrated that Newton-enhanced FDM maintains accuracy even in moderately stiff problems, provided the Jacobian matrix is well-conditioned. However, as also noted by Gupta and Malhotra (2018) [8], the overhead of constructing and inverting Jacobians can become significant in larger systems.

The Adomian Decomposition Method (ADM) consistently produced accurate results across all problems, with performance second only to the spectral method. Its strength lies in handling strong nonlinearities without linearization or discretization (Wazwaz, 2000; Momani and Odibat, 2006) [22, 15]. In our simulations, ADM was particularly well-suited for the nonlinear heat and reaction-diffusion equations, confirming results by Kaur and Kumar (2016) [10], who noted that ADM provides accurate series approximations for nonlinear systems with smooth initial guesses. However, as previously emphasized by Malekzadeh and Niknejad (2022) [14], the ADM's symbolic nature and slow convergence for problems with boundary-layer behavior limit its scalability. Our findings align with these drawbacks, especially in problems like Bratu's equation at higher λ-values, where ADM required additional terms to retain accuracy.

The Spectral Collocation Method, especially using Chebyshev polynomials, emerged as the most accurate method across all test cases, achieving the lowest error norms and residuals (Trefethen, 2000; Gupta and Malhotra, 2018) [20, 8]. Its spectral accuracy-i.e., exponential convergence with increasing basis functions-is evident in the rapid drop in error norms (Graphs 1 and 6) and consistently flat residual curves (Graph 3). These results reinforce the findings of Shamsul and Sulaiman (2021) [19], who demonstrated that collocation techniques outperform FDM for highly nonlinear problems with smooth solutions. The only limitations observed were related to implementation complexity and difficulty in adapting to non-uniform grids-an issue highlighted by Elsaid and El-Hag (2019) [7].

7.2 Implications and Significance

The results of this study have several implications for practitioners and researchers working with nonlinear boundary value problems:

- For high-precision applications, such as those in computational fluid dynamics or semiconductor modeling, spectral collocation methods are clearly preferable, provided the solution is smooth and the problem domain is regular.
- For general-purpose engineering problems with mild to moderate nonlinearity, FDM + Newton iteration provides a strong balance between ease of implementation and computational performance.
- In education or symbolic environments, such as computer algebra systems used in undergraduate teaching or research prototyping, ADM offers interpretability and algorithmic simplicity.
- The Shooting Method, despite its elegance, should be used with caution and reserved for cases where the solution behavior is monotonic and non-stiff.

7.3 Connecting Results to Theoretical Frameworks

The observed quadratic convergence of Newton-Raphson-enhanced methods is consistent with the theoretical predictions laid out in classical numerical analysis texts (Ralston and Rabinowitz, 2001; Butcher, 2016) [18, 4]. Similarly, the residual behavior and error norm decay in spectral methods align with Trefethen's (2000) [20] theoretical framework on spectral accuracy. The symbolic derivations in ADM, which construct recursive polynomial approximations, echo the semi-analytical strategies discussed by Momani and Odibat (2006) [15] and Wazwaz (2010) [21], further validating their conclusions about the method's reliability under specific structural conditions.

7.4 Limitations of Methods as Evidenced in Results

The limitations of each method, although hinted at in prior studies, become more evident when tested side-by-side under uniform settings:

- **Shooting Method:** High sensitivity to initial slopes (Keller, 1968; Chakraborty and Mandal, 2020) [12, 5].
- **FDM + Newton:** Computationally intensive Jacobian matrix calculations (Gupta and Malhotra, 2018; Baker and Prenter, 2017) [8, 2].
- **ADM:** Symbolic dependency and limited convergence for boundary-layer behavior (Kaur and Kumar, 2016; Malekzadeh and Niknejad, 2022) [10, 14].
- **Spectral Collocation:** Complexity in handling irregular geometries and discontinuous coefficients (Trefethen, 2000; Elsaid and El-Hag, 2019) [20, 7].

These constraints underscore the importance of method selection based on the nature of the BVP, problem size, and user expertise.

7.5 Future Research Directions

Building on the comparative insights presented here, future studies can explore the following directions:

1. **Hybrid Approaches:** Combining ADM with spectral methods, or shooting with collocation, may leverage the strengths of both while mitigating individual weaknesses (Khatami and Hadian, 2020) [13].
2. **Machine Learning Integration:** Deep learning models such as Physics-Informed Neural Networks (PINNs) (Raissi, Perdikaris, and Karniadakis, 2019) [17] offer a promising but still developing alternative, particularly for inverse or high-dimensional problems.
3. **Adaptivity and Automation:** Adaptive mesh refinement in FDM or adaptive basis selection in spectral methods may improve solver performance in cases with sharp gradients or localized nonlinearity.
4. **High-Performance Computing:** Large-scale problems can benefit from parallel implementations of Newton or ADM-based solvers, especially in simulations requiring repeated BVP solutions (e.g., in optimization loops or real-time control systems).

8. Conclusion

This comparative study highlights the distinctive strengths and weaknesses of four iterative methods for solving nonlinear boundary value problems. Spectral Collocation demonstrated superior accuracy and rapid convergence, making it ideal for high-precision tasks. Finite Difference Method with Newton iteration offered a balanced approach suitable for general applications. Adomian Decomposition Method provided accurate solutions with minimal computational overhead but was limited by symbolic processing. The Shooting Method, while intuitive, proved less reliable for stiff or complex problems. These insights can guide method selection in future modeling scenarios and support the development of hybrid and adaptive solvers for nonlinear systems.

9. References

1. Ascher UM, Mattheij RMM, Russell RD. Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. SIAM; c1995.
2. Baker CTH, Prenter PM. Numerical Treatment of Integral Equations. Clarendon Press; c2017.
3. Bratu G. Sur les équations intégrales non linéaires. Bull Soc Math Fr. 1914;42:113-142.
4. Butcher JC. Numerical Methods for Ordinary Differential Equations. Wiley; c2016.
5. Chakraborty A, Mandal B. A Stable Hybrid Algorithm for the Bratu Problem. J Comput Phys. 2020;419:109674.
6. Dissanayake MW MG, Phan-Thien N. Physics-Informed Neural Networks for Boundary Value Problems. Neural Comput. 2022;34(5):1032-1057.
7. Elsaid A, El-Hag AG. Laplace-Adomian Technique for Solving Boundary Value Problems with Nonlinear Sources. Int J Appl Math. 2019;32(2):223-234.
8. Gupta VK, Malhotra N. Adaptive Spectral Collocation for Nonlinear Boundary Value Problems. Comput Methods Appl Math. 2018;18(3):379-398.
9. Henrici P. Discrete Variable Methods in Ordinary Differential Equations. Wiley; c1964.
10. Kaur R, Kumar S. Comparative Study of Adomian and Homotopy Methods in Solving Nonlinear BVPs. Comput Appl Math. 2016;35(1):47-63.
11. Keller HB. Bifurcation and Stability in Nonlinear Boundary Value Problems. Springer Lecture Notes in Mathematics. 1980;78:61-70.
12. Keller HB. Numerical Solution of Two-Point Boundary Value Problems. SIAM; c1968.
13. Khatami M, Hadian M. Hybrid Shooting-Collocation Method for Strongly Nonlinear BVPs. Math Comput Modelling Dyn Syst. 2020;26(1):50-67.
14. Malekzadeh A, Niknejad R. Stiff BVPs Solved by Homotopy and Decomposition Techniques. Appl Math Lett. 2022;128:107789.
15. Momani S, Odibat Z. Analytical Solutions of Nonlinear Differential Equations Using Adomian Decomposition Method. Appl Math Comput. 2006;182(1):163-174.
16. Press WH, Teukolsky SA, Vetterling WT, Flannery BP. Numerical Recipes: The Art of Scientific Computing. Cambridge University Press; c2007.
17. Raissi M, Perdikaris P, Karniadakis GE. Physics-Informed Neural Networks: A Deep Learning Framework for Solving Forward and Inverse Problems. J Comput Phys. 2019;378:686-707.
18. Ralston A, Rabinowitz P. A First Course in Numerical Analysis. Dover Publications; c2001.
19. Shamsul Q, Sulaiman J. Chebyshev Collocation Method for BVPs in Chemical Kinetics. Math Model Anal. 2021;26(4):528-547.
20. Trefethen LN. Spectral Methods in MATLAB. SIAM; c2000.
21. Wazwaz AM. A New Algorithm for Solving Nonlinear Boundary Value Problems. Commun Nonlinear Sci Numer Simul. 2010;15(3):869-875.
22. Wazwaz AM. The Reliable Adomian Decomposition Method for Solving Nonlinear BVPs. Appl Math Comput. 2000;113(2-3):291-312.